Every organism on Earth responds to four major cycles: the solar and the lunar day, the synodic month, and the year. We all dance to these primary rhythms. This book reveals the poetic cosmology that lies within the cycles of the Sun and Moon as seen from the Earth. Experience the uniquely connected universe that Robin Heath presents: solstices, eclipses, and equinoxes all concisely described and illustrated. The timeless beauty of our universe has never been more apparent or intriguingly simple.

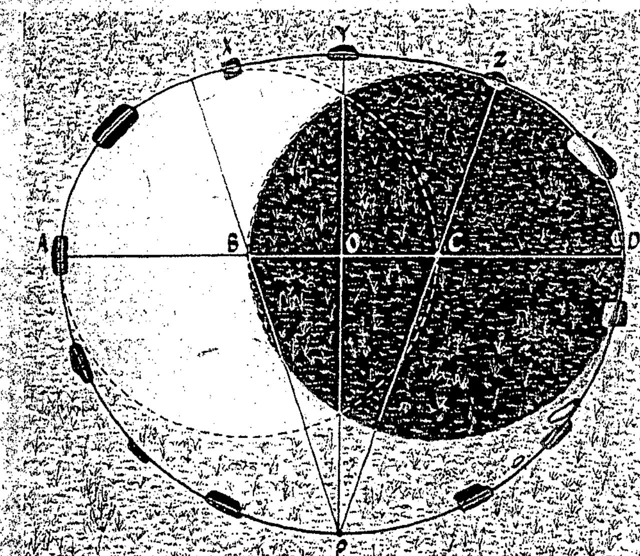
ISBN 0-8027-1381-5

9 780802 713810

WOODEN BOOKS

SUN, MOON & EARTH • Heath • WOODEN

SUN, MOON, & EARTH



obin Heath

Originally published in Wales by Wooden Books Ltd. in 1999; first published in the United States of America in 2001 by Walker Publishing Company, Inc.

Published simultaneously in Canada by Fitzhenry and Whiteside, Markham, Ontario L3R 4T8

Printed on recycled paper.

Library of Congress Cataloging-in-Publication Data

Heath, Robin.
Sun, moon, & earth / written and illustrated by Robin Heath.
p. cm.
Originally published: Wales : Wooden Books, 1999.
ISBN 0-8027-1381-5 (alk. paper)
1. Archaeoastronomy. I. Title: Sun, moon, and earth. II. Title.
GN799.A8 H47 2001
520--dc21 2001026012

Printed in the United States of America

4 6 8 10 9 7 5

SUN, MOON, & EARTH

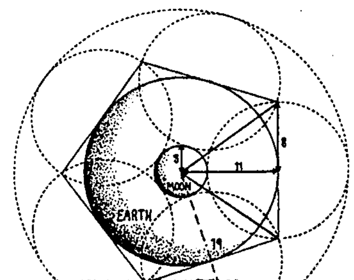
written and illustrated by

Robin Heath

WOODEN BOOKS

Walker & Company
New York

SUN, MOON, & EARTH

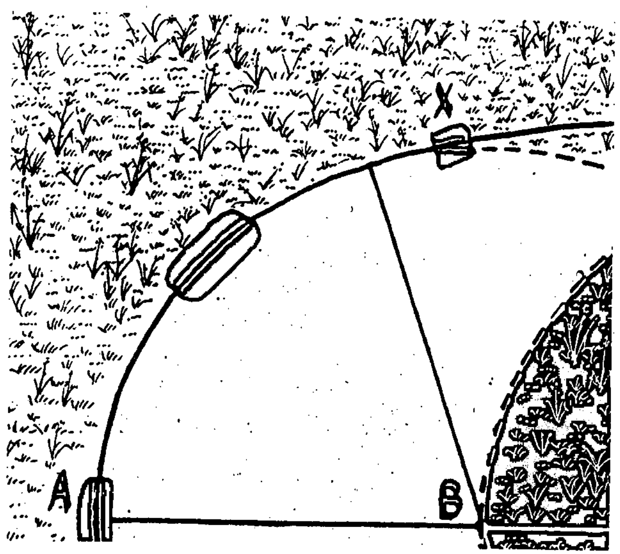


For all the new children

Special thanks go to my brother, Richard, for being a vital part of the early research. Many thanks to John Martineau.

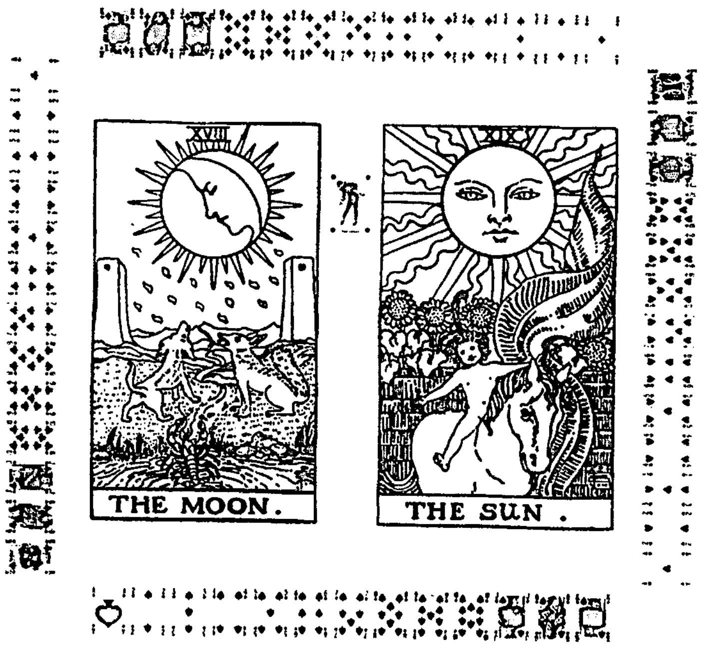
Note: Some descriptions in this book assume you are living in the northern hemisphere.

The title page shows the Venus of Laussal, circa 18,000 B.C. A clear message survives the aeons, confirming ancient human knowledge of the link between the Moon and the human reproductive cycle. Thirteen notches on a crescent horn link astronomy with human culture.



Above: Bar Brook, Derbyshire. A typical type-B flattened stone circle, 4,500 years old and revealing a subtle cosmology and metrology.

CONTENTS



ot card numbers eighteen and nineteen—the Moon and the Sun—framed by a ying card set: 4 suits of 13 cards, representing the 52 weeks in the year. Adding the numerical values of each suit (the sum of 1 to 13) yields 91, the number of s per season. All four seasons then total 364 days, the joker completing the year 365. An ancient aide memoire *for the calendar.*

INTRODUCTION

In the frenzied attempt to better understand and control the material universe, our present culture has strayed so far from simplicity and beauty that we are often startled or afraid when it reveals itself.

The modern system of ideas we call *science* has dispensed with the poetic and broadly fails to see the subtle connecting strands of meaning woven into the web of life. In addition, science is today shackled to commerce, also blind to such things, and thus we have "the blind leading the blind." As if that wasn't enough, today's high priests of science also inform us which interpretations of the cosmos are valid and which are not. Some questions are just not to be asked anymore, let alone answered.

This little book reveals a poetic cosmology that lies within the cycles of the Sun and Moon, as seen from Earth. It is found to be supremely rational, so simple and elegant that no priestly intermediary is needed to interpret, censor, or intervene.

All the mathematics given here may be verified by those of little faith with a simple calculator and the mind of an inquisitive teenager.

St. Dogmaels, 2001

SEARCHING FOR PATTERNS

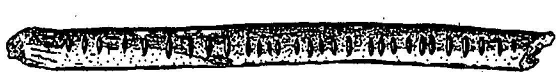
finding order in the cosmos

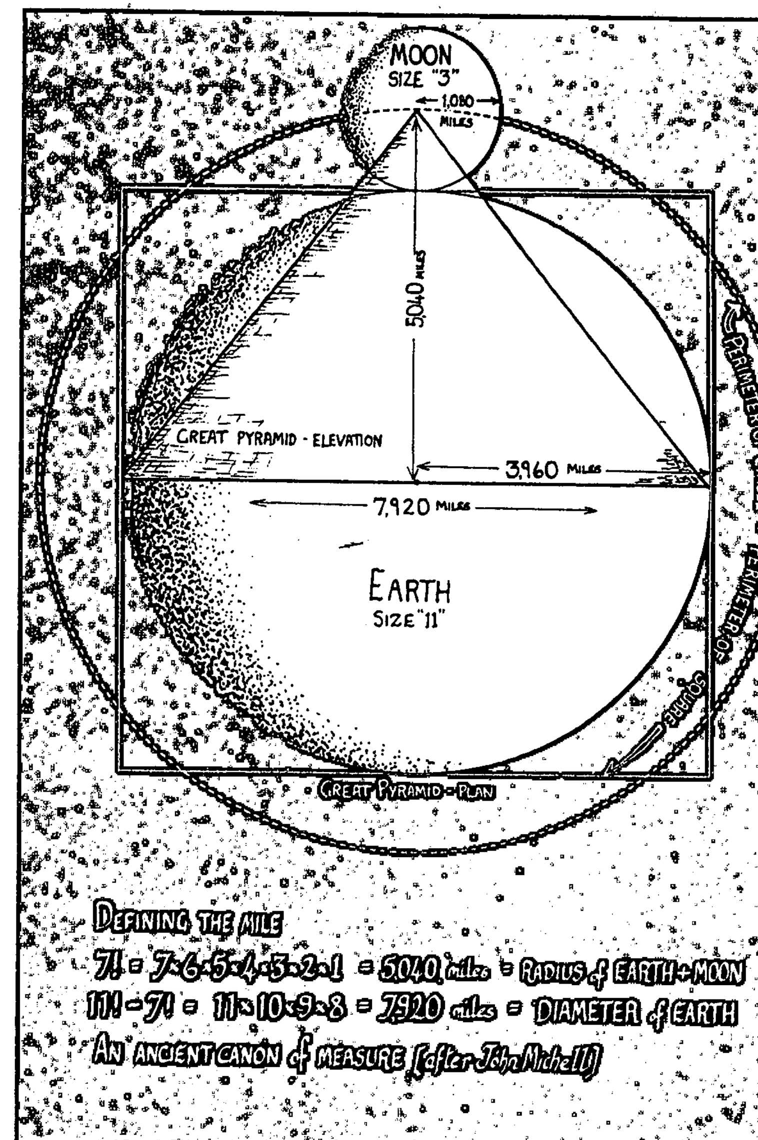
The first systematic observations of the Sun and Moon are shrouded in the mists of prehistory. Scored bones from 40,000 B.C. (*below*) display lunar number cycles while the famed *Venus of Laussal* (*title page*) links the Moon with the number thirteen.

Repeated cycles such as full moons, eclipses, and planetary conjunctions revealed a cosmology to ancient astronomers that was both numerical *and* geometrical, and which imbued creation with order and meaning—"God is a geometer." The delphic adage "as above, so below" suggests that cosmic patterns are reflected in earthly life, becoming a source of revelatory information.

The Great Pyramid (2480 B.C.) epitomizes this approach. Built to the points of the compass, with passageways aligned to stars, its base and height fit the "squared circle" of Earth and Moon.

This archaic approach to cosmology is today discarded as worthless, and has been replaced by modern astronomy. Yet most people know almost nothing about the Sun, Moon, and Earth system, despite our total dependence on its rhythms. This book will gently put that right, and reinvoke something of the spirit of the old sciences.





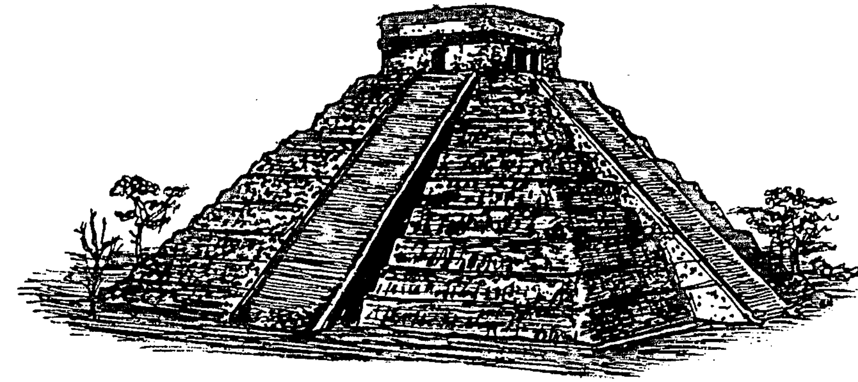
Some Early Solutions

from megaliths to the Maya

Skywatching is an ancient art. Stone circles date from 3000 B.C., aligned megaliths even earlier. The Egyptians were using accurate surveying and a precise metrology for both sky and Earth. The Great Pyramid enshrined its date of construction through astronomical alignments to fixed stars. The Sumerians recorded astral cycles from 2200 B.C. and later defined the 24-hour day and 360-degree circle. Chaldean and Chinese astronomers knew of the Saros eclipse cycle (*page 28*). Various calendars were in use.

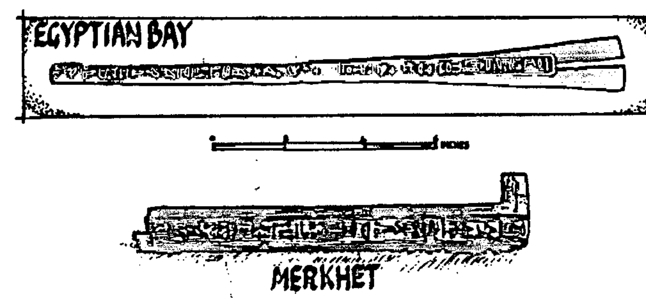
From 600 B.C., the Greeks inherited this ancient wisdom. Eratosthenes measured the size of the Earth and Eudoxus devised a solution for the complex motion of the Moon. In the fourth century B.C., the nineteen-year cycle of Sun and Moon was described by Meton. The Romans gave us our modern calendar in 45 B.C.

When the Empire collapsed around A.D. 500 the Arab world kept the torch of learning burning as Europe sank into the Dark Ages. Following the Crusades this material returned, seeding the Renaissance in Europe. Copernicus showed that the Earth orbited the Sun, while Galileo's telescope revealed moons orbiting other planets. Kepler published the three laws of planetary motion in the early seventeenth century, when Newton used data about the Moon to quantify his universal laws of motion and gravity in 1687, thereby spawning our modern world. In the next century, Harrison's chronometer greatly improved timekeeping and navigation.



El Castillo, Chichén Itzá

4 flights of 91 steps totaling 364 plus the high altar—365.



~ Early Surveying Instruments from Egypt ~

These instruments belonged to an "Hour Priest" of the twenty-sixth Dynasty, circa 1000 B.C.

THE SUN

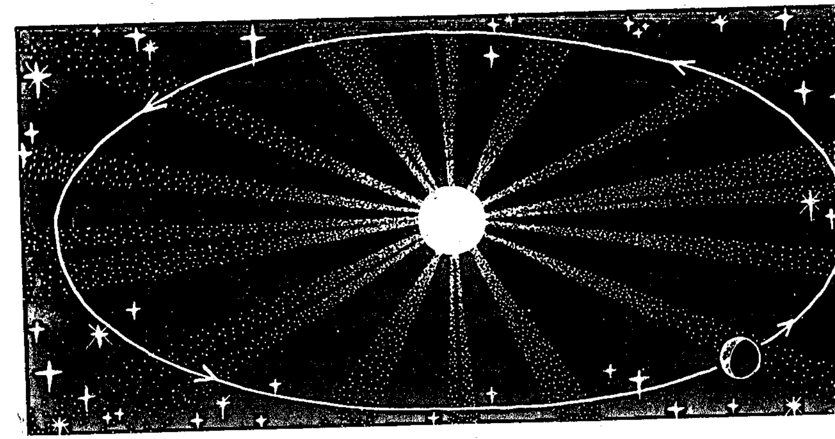
the day and the year

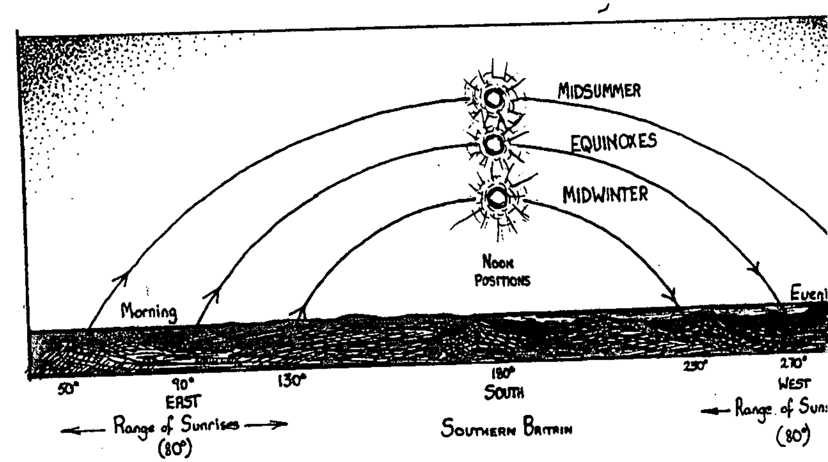
Each day, the Sun appears to rise from an easterly direction, traces a clockwise arc across the heavens, and then sets toward the west, disappearing for the dark time we call night. This cycle repeats perpetually; it is the diurnal rhythm, called, more simply, a day.

Today we are taught that what we see is caused by the daily rotation of a spherical Earth orbiting the Sun. Thereafter, like The Fool on the Hill, we "see the Sun going down, while the eyes in our head see the Earth spinning round." Each day, the Sun appears to move about a degree counterclockwise (eastward) with respect to the fixed stars. Thus the solar day, to which we set our clocks, exceeds the *sidereal* (star) day by 3 minutes and 56 seconds.

The axial tilt of the Earth (*page 9*) causes the Sun to rise and set each day at different positions on the horizon. Only at the summer and winter *solstices* (see page 8) is this daily change in the Sun's rise and set positions reduced to zero, at their extreme *standstill* positions. Subsequent sunrises and sunsets gradually reverse back along the horizon, the span being dependent on the latitude of the observer (*opposite, bottom*). This is the rhythm of a year.

The Earth's solar orbital period is 365.242199 days. Our "solar" calendar of 365 days keeps pace by adding regular leap-year days, one every four years (except once every four hundred years), and the odd second or two.





Solstices and Equinoxes

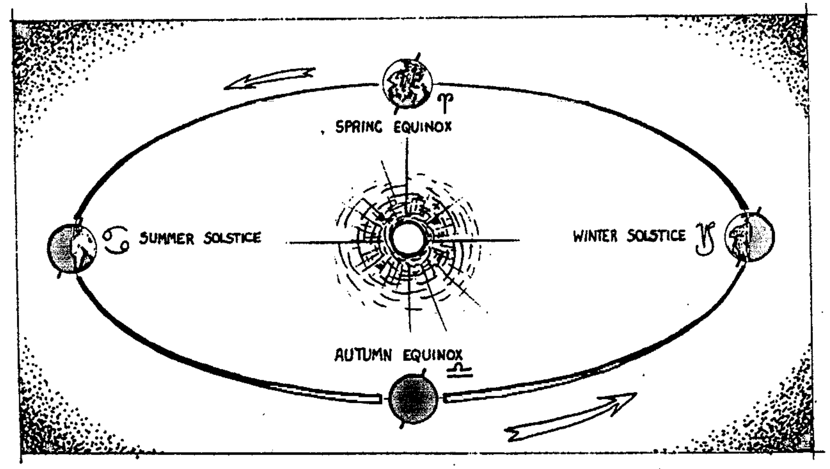
the four natural divisions of the year

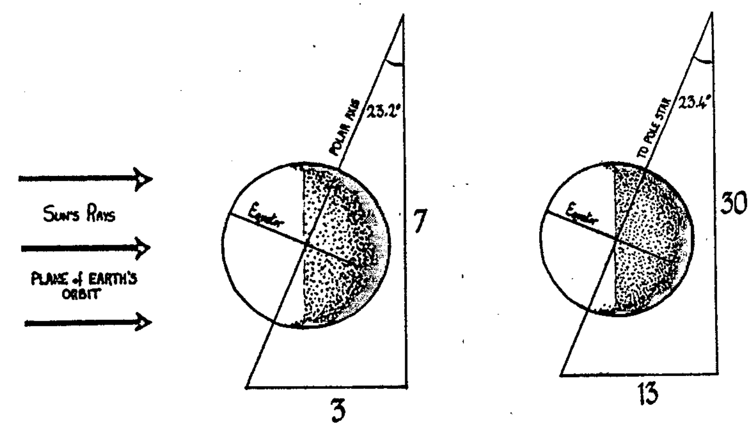
In between the two solstices, the longest and shortest days in the year (normally June 21 and December 22), lie the two equinoctial periods in the spring and the autumn. The *equinoxes* (March 21 and September 23) deliver equal lengths of day and night everywhere on the planet, with the Sun rising exactly due east and setting exactly due west, on a level horizon.

These equinoctial dates are accompanied by the maximum rate of change in the length of the day. In temperate latitudes, this creates the impression that the year is divided into two distinct halves, a light, warm summer half and a dark, cold winter half. During the summer half the Sun rises and sets north of an east-west line; in the winter half always south of it.

The solstices and equinoxes naturally divide the year into four quarters, defining the four seasons. Each season is 91 days in length (*see page 5 and opposite page 1*), caused by the tilt of the Earth on its own axis (currently $23\frac{1}{2}°$ with respect to its orbital plane). This angle may be constructed using a right triangle, base 13 and height 30, or more approximately 3 and 7.

The "cross-quarter" days, halfway between equinoxes and solstices, are still celebrated as the Celtic festivals of *Samhain* (November), *Imbolc* (February), *Beltane* (May Day), and *Lughnasadh* (August). The Earth orbits the Sun at the incredible speed of 66,666 miles per hour and at a distance of 108 solar diameters.





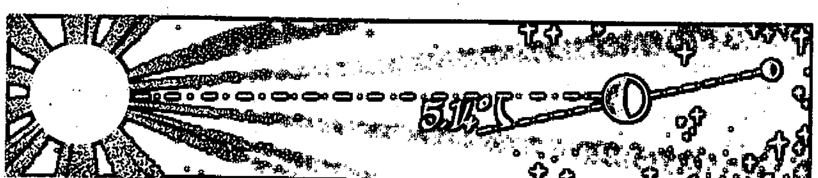
The Moon

the goddess of the night

Although apparently lifeless, the Moon greatly affects life on Earth. The fluctuating monthly rhythm of reflected light, the twice daily ebb and flow of the tides, and many natural cycles are all essentially locked into the lunar phases as, uniquely, is the reproductive cycle of humankind. The Moon is associated with women and the number 13, perhaps because the Moon moves 13 degrees a day and orbits the Earth 13 times in one year. People see a man in the moon or sometimes a hare, owl, swan, or lady.

At an average distance of 240,000 miles, the Moon is our nearest neighbor. Its radius is 1,080 miles compared with that of the Earth at 3,960 miles, a ratio of 3:11. However, the Moon is not spherical, and the Earth's gravity always pulls the larger hemisphere toward us. The Moon thus has its "dark side," which we never see, but which paradoxically becomes fully lit each new moon.

The Moon's orbital plane is tilted to that of the Earth (*below*). Periodically, this enables eclipses to occur and, at higher latitudes, every 18.618 years, causes wild monthly fluctuations in the altitude of the Moon, and a maximum angular range of rising and settings.



The Moon's Two Rhythms

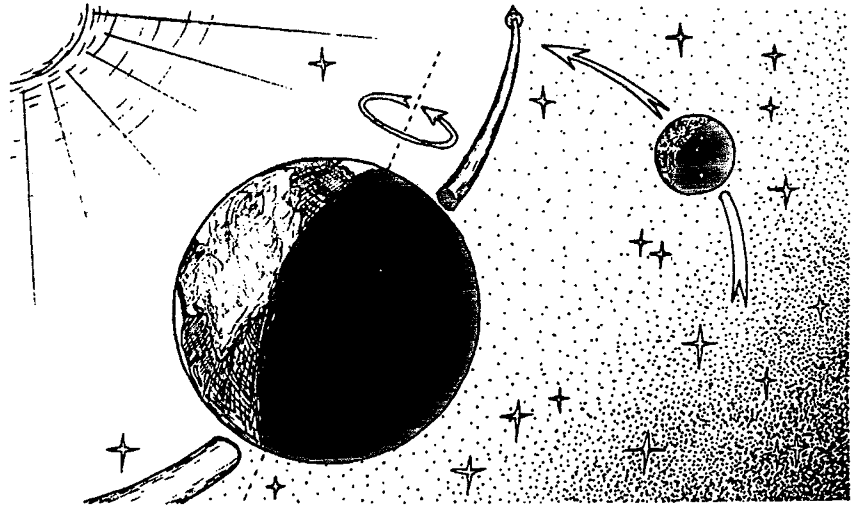
sidereal and synodic months

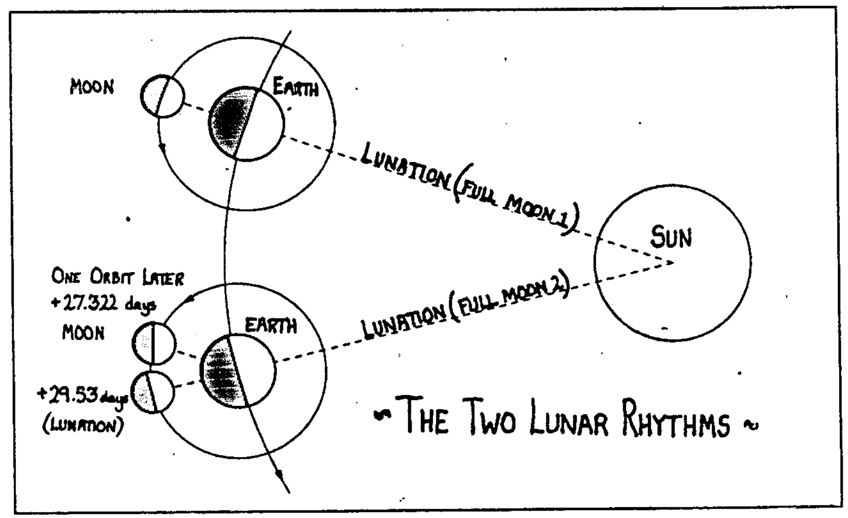
Observe the Moon for a short while and you will discover one of its rhythms — it moves past the fixed stars quite rapidly, taking about an hour to cover the distance of its own diameter. In one day, it covers 13 degrees, thereby taking slightly less than 28 days to return to the same stars. This is the *sidereal* month of 27.322 days, or 27 days, 7 hours, 43 minutes, and 11.51 seconds (approximately $27\frac{18}{56}$ days).

Three principle rotations are now defined: the *day* (Sun-Earth), the *sidereal month* (Moon-Earth-stars) and the *year* (Sun-Earth-stars). All three are shown opposite. But there is a fourth rotation, the lunar phases or *lunation cycle*, the time between full moons, which is truly Sun, Moon, and Earth. Because the lunar phases are visible all over the Earth, the lunation cycle is the prime lunar rhythm. It is also called the *synodic month* or, more simply, the *lunar month.* It takes 29.53059 days to complete, or 29 days, 12 hours, 44 minutes, and 2.37 seconds (approximately $29\frac{43}{81}$ days).

The Earth completes about a thirteenth of its annual journey around the Sun during one sidereal month, thus the lunar phases have to "catch up," as shown in the illustration (*opposite, bottom*), this taking an extra 2.21 days over and above the sidereal month.

There are 13.368 sidereal months and 12.368 synodic months in the year. The fractional part is very close to seven-nineteenths.





THE TICK-TOCK OF THE MOON

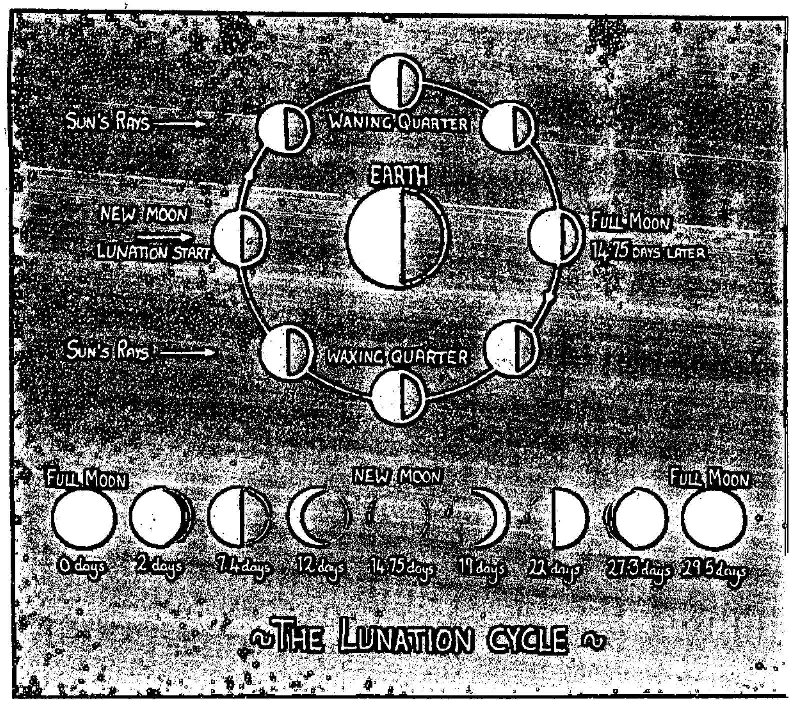
the lunation cycle or lunar month

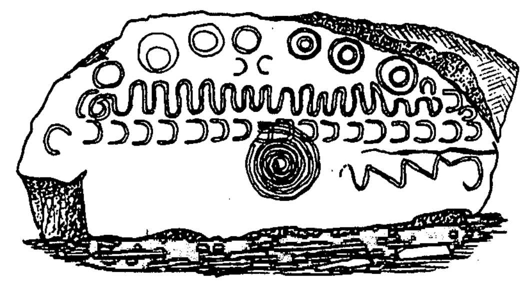
The Moon begins her monthly phases acting as consort to the Sun, appearing as a sliver of silver to his left. This is the *new moon*, shaped like a reversed *C*. Each successive day finds the Moon belonging more and more to the night sky as the *waxing* phases increase the crescent to a *quarter*, *gibbous*, and then *full* moon, taking about thirteen days to complete.

Only when full does the Moon "escape" the Sun, becoming entirely nocturnal and reflecting the maximum silvery light down onto the night landscape. The *waning* cycle then progressively delivers the Moon back into the daytime skies as it leads the Sun, setting later and later in the day until, again after about thirteen days, it only becomes separate from the Sun just before dawn, glimpsed as a tiny *C*-shaped crescent to the Sun's right.

The Moon then disappears for about three days, lost in the light of the Sun at the new moon. This whole cycle is called the *lunation cycle*, lunar, or *synodic*, month; it is the time between full (or new) moons, and takes an average 29.53059 days to complete.

An inscribed 5,000-year-old curbstone at Knowth, in the Boyne valley, Ireland, displays what appears to be a representation of the lunation cycle. The impressive spiral correctly covers the three days of new moon, and 15 days later the full moon is marked ") (" within a 29-based motif. The "serpent" enclosed by this lunation motif has 30 turns, 29 $\frac{1}{2}$ being the average between the two.





KNOWTH K53

THE LUNAR DAY

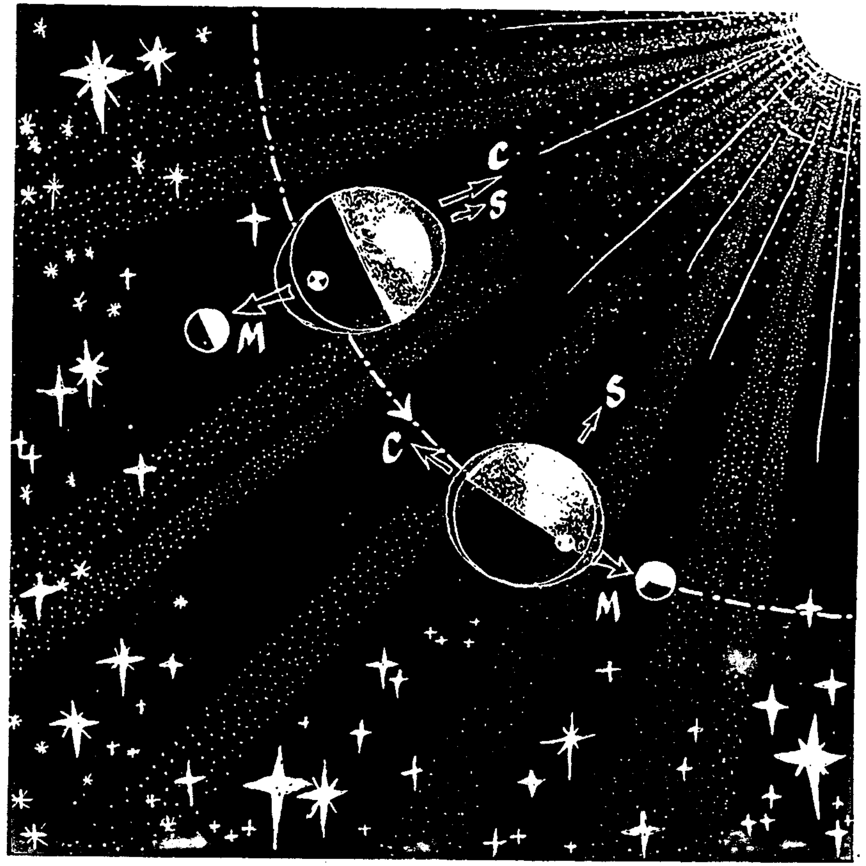
tides of lunar time

Anyone who lives by the sea witnesses the inexorable silent power of the Moon, whose invisible claws draw the tides up and down the beach twice a day. Tides are not just limited to the oceans—the Earth's atmosphere above our heads and even its crust beneath our feet rise and fall to this lunar rhythm. The highest *spring* tides occur two or three days following a new or full moon. The low-range *neap* tides occur two days after a waxing or waning quarter moon.

The *lunar day* is the time between consecutive moonrises, the Moon rising an average of 52 minutes later each day. There are exactly two tides each lunar day, each one retarded by an average of 26 minutes every 12 hours. Tides are synchronized to the lunar day and therefore to the Moon's position in the sky. High tides will always occur at the same two positions of the Moon in the sky at any given location, these being opposite each other (one position is always beneath the horizon). A practical tidal indicator is shown on page 47.

There are $28\frac{1}{2}$ lunar days and therefore 57 tides (3 x 19) in each lunation cycle. In isolation tanks, human bodily rhythms eventually transfer from the solar to the lunar day.

The Earth and Moon form a huge dumbbell in space, with their center of rotation located about 1,000 miles beneath the Earth's surface (*shown as a small checkered circle opposite*).



The Moon's gravitational pull (M) lifts the oceans on the side of the Earth facing the Moon. On the opposite side, centrifugal force causes a similar effect (C), because the center of mass (and so of rotation) of the Earth-Moon system does not lie at the center of the Earth. The Sun also pulls at the oceans (S), and according to the phase of the Moon adds or subtracts to and from the height of the tide. "Spring" tides occur near full and new moons (upper), the lesser "neap" tides at the quarter-moons (lower). The monthly ratio between the heights of spring and neap tides is 8:3.

Sun, Moon, and Landscape

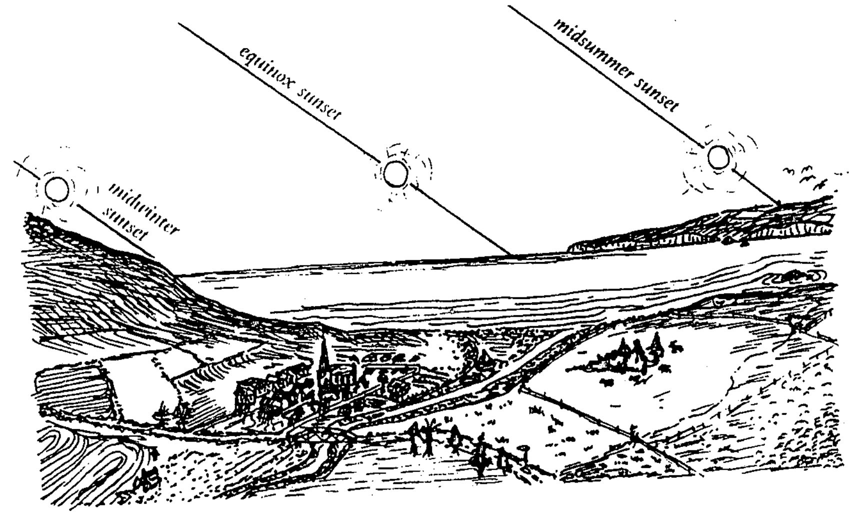
the moon as mirror of the sun

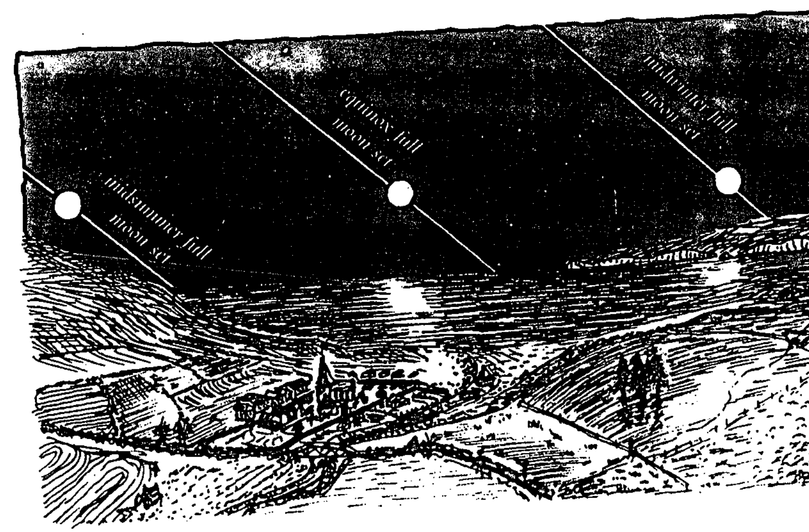
Each month the Moon more or less copies the entire annual range of rises and falls undertaken by the Sun in a year.

The Moon rises highest in the sky each month when it is found near Betelgeuse, in Orion. Its most northerly risings and settings occur then. At extreme latitudes (above 60°) a midwinter full moon may become *circumpolar* and not set for a few days. The most southerly risings and settings occur when the Moon is found near Antares, in Scorpio. At extreme latitudes, for example in Finland or northern Canada, the full moon may not be visible during midsummer, especially above the arctic circle, where the Sun becomes circumpolar.

The full moon is brightest and highest at midwinter, copying the motion of the midsummer Sun. The midsummer full moon correspondingly behaves like the midwinter Sun, remaining low in the dusky sky. Thus the full moon mirrors the Sun at the opposite point in the calendar, and like a true mirror it fully reflects the Sun's light.

This reciprocation mysteriously extends into the numbers, for 1 ÷ Sun = Moon, and 1 ÷ Moon = Sun! 1 ÷ 365.242 = 0.0027379, which in days is 3 minutes and 56 seconds, the difference between sidereal and solar days, while 1 ÷ 27.322 = 0.0366, which in days is 52 minutes, the difference between lunar and solar days. It is fun to ask an astronomer why.





The Moon's Nodes

the path of the moon crosses that of the sun

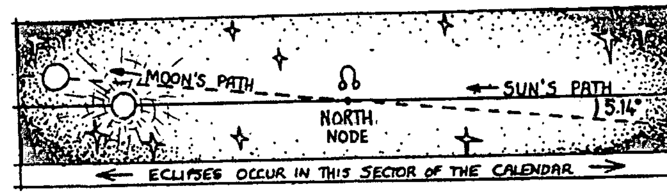
The orbit of the Moon is tilted with respect to that of the Earth by an angle of 5.14° (*see page 10*). The effect is that the Moon travels above the ecliptic (the apparent path of the Sun around the zodiac) for about half the sidereal month, and travels beneath it for the other half.

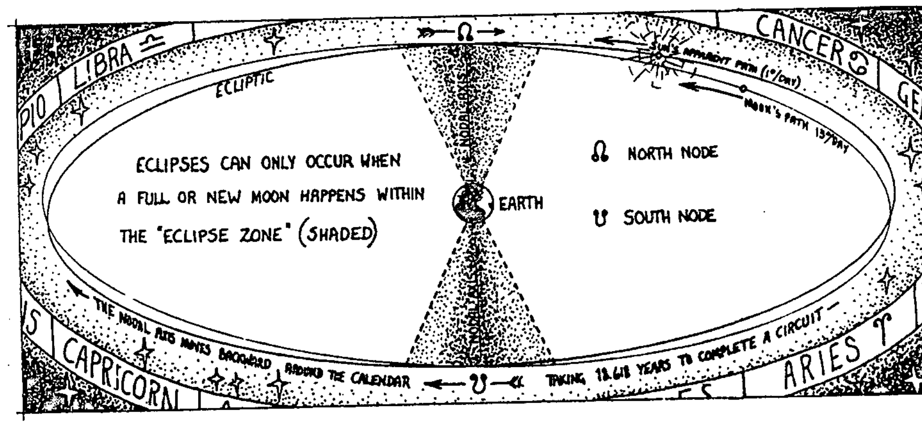
The two places where the Moon crosses the ecliptic each month are called the *lunar nodes,* and they always lie opposite each other. The two smaller illustrations opposite show these crossing points as observed from the Earth—but in truth, they are invisible! Eclipses only happen when a full or new moon occurs within $12\frac{1}{2}°$ or $18\frac{1}{2}°$, respectively, of the nodes; total eclipses when the alignment is almost exact. These are the *eclipse limits* for lunar and solar eclipses.

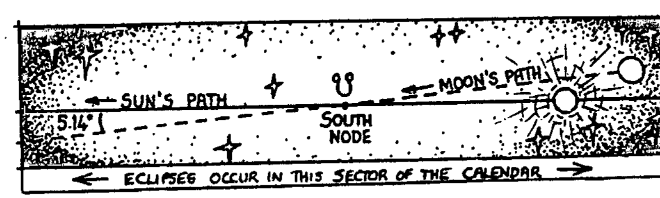
The axis of the nodes moves backward around the calendar, taking 18.618 years (6,800 days) to complete a circuit. It moves 19.618 days per year. To the ancients the nodes were thought of as the head and tail of a huge celestial dragon that swallowed the Moon or Sun during an eclipse. The nodal period is still known as the *Draconic year.*

The Sun meets a node every 173.3 days (an *eclipse season*); it meets a particular node after two of these periods have elapsed, this both defining and completing the *eclipse year* of 346.62 days.

Is it not the strangest thing that 346.62 = 18.618 x 18.618?







THE "BREATH" OF THE MOON

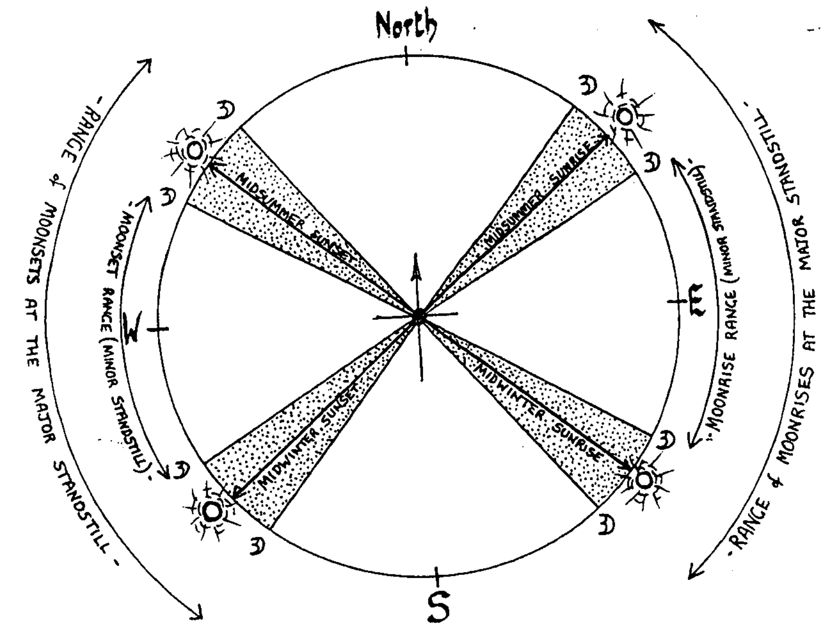
major and minor standstills

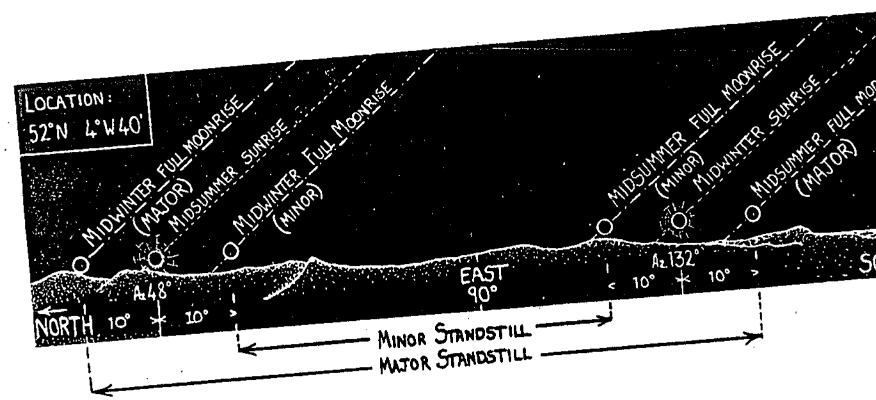
The monthly extreme northerly and southerly risings and settings of the Moon gently "breathe" in and out either side of the Sun's extreme solsticial positions, taking one nodal period to complete the "breath." This greatly alters the possible maximum rising and setting positions of the Moon each month with respect to the solsticial positions of the Sun. There are thus eight limiting "lunstice" positions, four for risings and four for settings (*opposite, top*).

The distance of these extreme positions of the lunstice from the solstice position is dependent on the latitude of a location. In southern Britain, they occur more than eight degrees of either side of the solstice positions (*opposite, bottom*). These extreme stations of the Sun and the Moon drew the attention of neolithic astronomers who made alignments of stones in their honor.

At the *major standstill*, the Moon describes her wildest monthly swings of rising and setting, gyrating to her highest- and lowest-ever paths across the sky, all within one sidereal month. At the *minor standstill*, 9.3 years later, the Moon calms down and the range always lies inside the solsticial positions.

In contrast to the Sun, Moon, and planets, the stars rise and set at exactly the same place along the horizon for hundreds of years irrespective of the season and time of day or night.





Eclipses

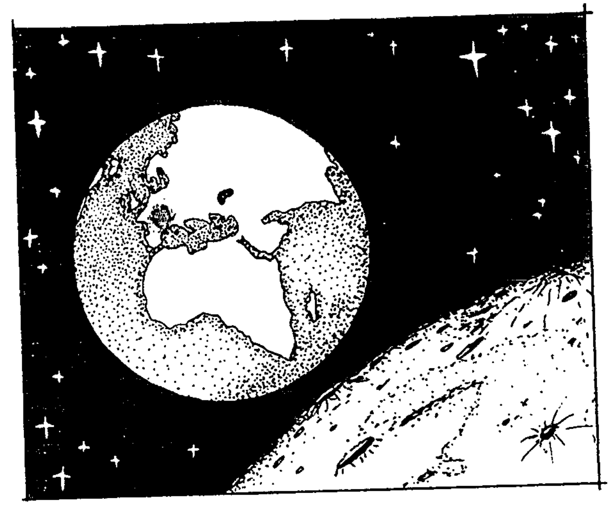
cosmic attention seeking

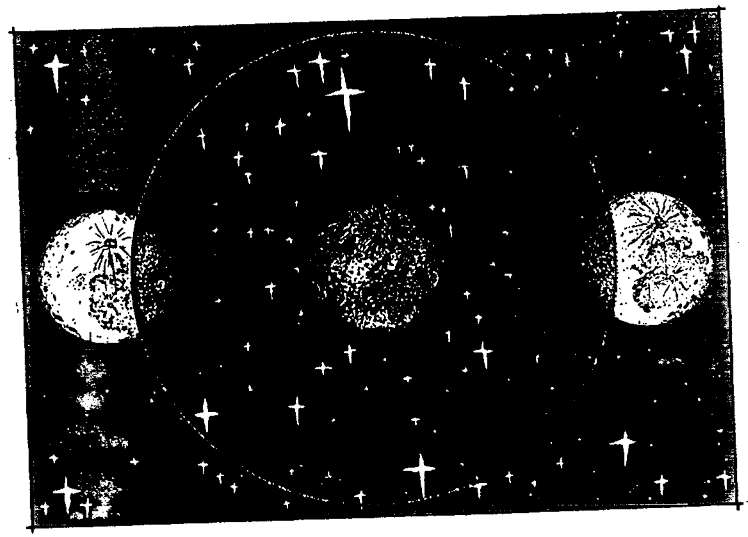
By what almighty coincidence do the disks of Sun and Moon appear the same size to us earthlings? The Sun is four hundred times larger than the Moon, yet four hundred times farther away. The distance of the Moon from the Earth is just over thirty Earth diameters. Total solar eclipses could never occur if the Moon's orbital distance was changed by just one Earth diameter.

Total solar eclipses instill an elemental awe in us, with a sudden brief reversal from light into darkness, after which "dawn" returns *from the west*, at over 2,000 miles per hour! Lunar eclipses are gentler and longer and simulate a whole lunation cycle in just a few hours.

Solar eclipses occur when the new moon passes directly between Sun and Earth. They can only be seen during the daytime, the area of totality tracing a narrow smudge of blackness across the Earth. Totality never lasts longer than seven minutes at any one location (*opposite, top*). During lunar eclipses the full moon passes from right to left through the Earth's shadow, its reflected light extinguished for several hours (*opposite, bottom*). Lunar eclipses are visible to all on the night side of the Earth.

As the angle between Sun and node increases, total eclipses decrease and become partial (*see page 30*). Beyond 18.5° no eclipse can occur. There can be up to seven eclipses in any one year, and solar eclipses are the more common, by the ratio √2:1.





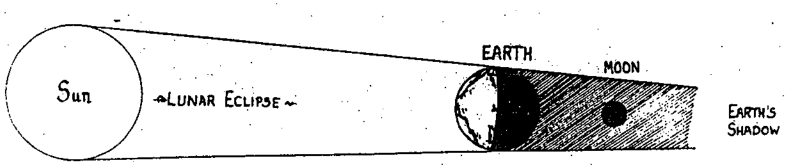
LUNAR ECLIPSES

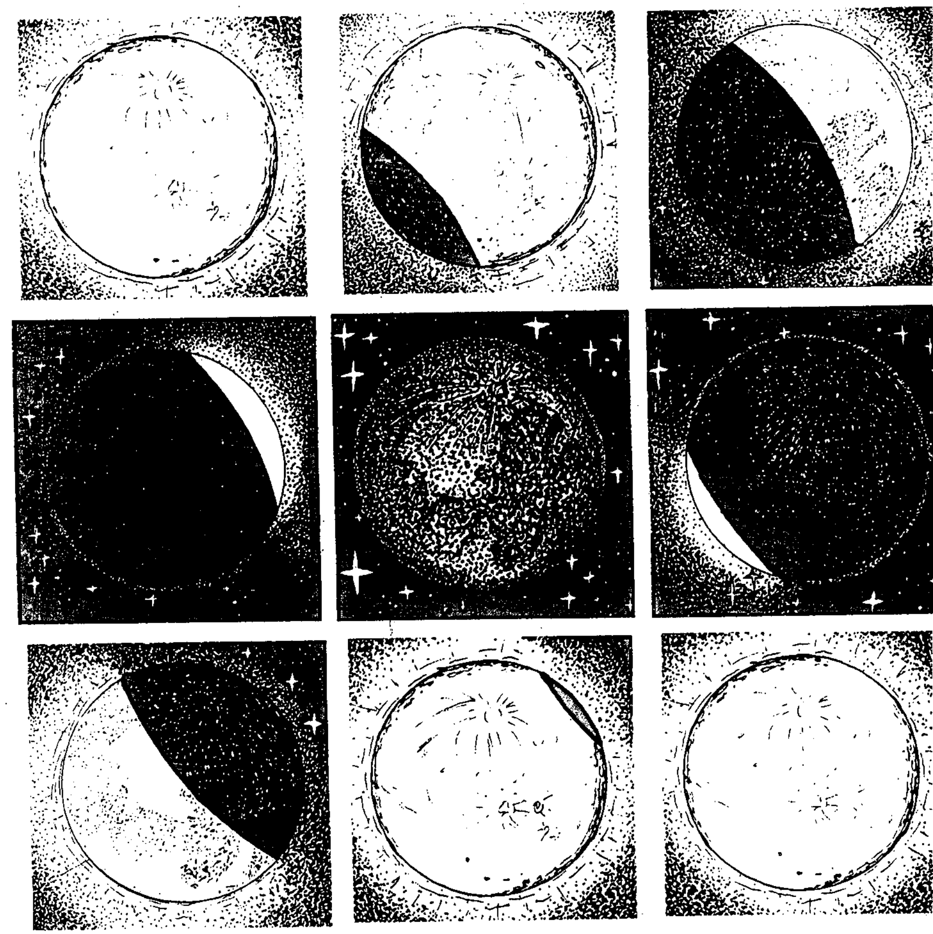
studies in light and darkness

A lunar eclipse is a striking phenomenon. The reflected light of the full moon greatly diminishes the light from the stars, and during the eclipse a curious and beautiful effect unfolds. As the full moon enters the Earth's shadow cone (*page 25, bottom*), the Moon's face darkens and the night sky radically alters its appearance, becoming brilliantly peppered with many more stars than were previously visible. This effect is also shown opposite, where a satisfying diagonal symmetry may also be seen.

During the period of totality, the Moon often takes on a remarkably beautiful coppery color within the starry firmament. Also stirring is the curve of the Earth's shadow as it draws across the lunar orb. It confirms that our planet is about three times larger than the Moon and spherical in shape.

Before 2500 B.C., Megalithic astronomers in northwestern Europe appear to have observed a tiny variation in the 5.14° tilt of the Moon's orbit ($\frac{1}{6}$° with period 173.3 days) in order to predict eclipses. Their observatories still exist, mainly in Scotland. What were our ancestors up to?





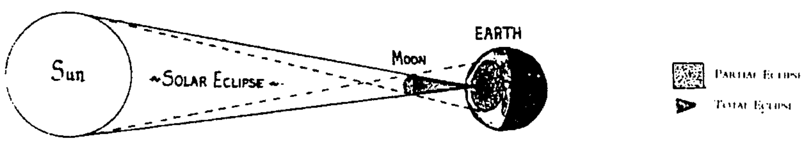
SOLAR ECLIPSES AND THE SAROS

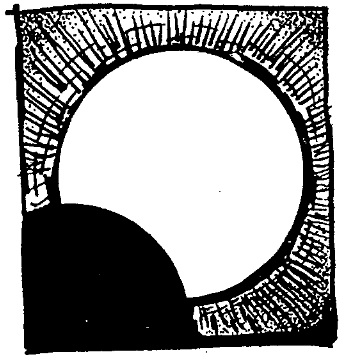
the 18-year cycle of eclipses

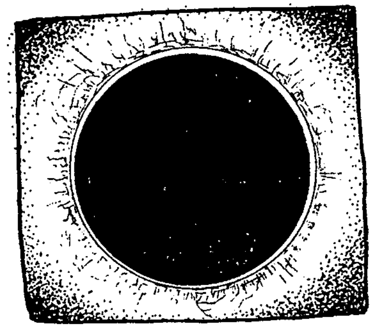
There are three types of solar eclipse: *partial*, *annular*, and *total* (*all shown opposite*). These are produced by variations in orbital distances and nodal offset at *syzygy*, the barely pronounceable term for a full or new moon (Sun, Moon, and Earth in line).

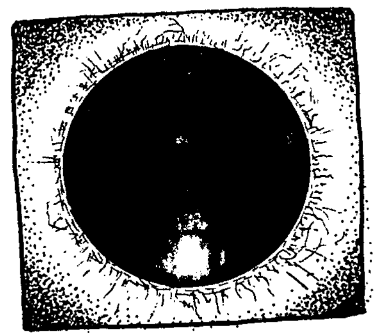
Any particular eclipse is a member of a family, consecutive individuals of which display similar characteristics. A famous family is the *Saros cycle*, of 18 years and 11 days, 223 lunations or 19 eclipse years. A Saros cycle evolves and decays over about 1,300 years (solar eclipses) and 800 years (lunar eclipses). At any given time an average of 42 Saros families of solar eclipses, and 27 of lunar eclipses, are evolving, each delivering about 70 and 45 individuals respectively over its lifetime. The Saros was used by the ancient Chaldean astronomers to accurately predict eclipses.

The eclipse year (346.62 days) equals 11.738 lunations. Divide 19 eclipse years (the Saros) by 11.738 and you get 1.6186, almost *phi*, the *Divine Proportion*. Because the eclipse year is 18.618 x 18.618 days, the Saros may be written as 19 x 18.618 x 18.618 days (to 99.99%). Mysteriously, 19 lunations is *phi* eclipse years!





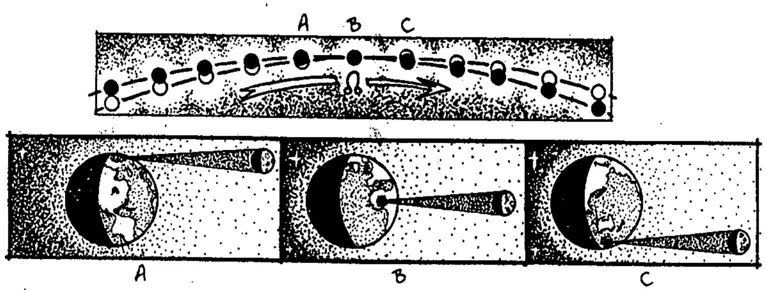




Saros Patterns
a trilateral evolution

All Saros eclipse families first make an appearance at one of the poles, gradually evolving into equatorial regions before eventually dying out at the other pole (*below, bottom*). The time difference between the Saros and 19 eclipse years (0.46 days) causes each new member to be displaced about half a degree farther west with respect to the nodes. Thus the family takes about 36 Saros cycles (650 years) to reach the node (B) and thereafter departs from it in the same time, slowly dissipating (*below, top—every 7th Saros shown*).

The patterns made by these metamorphosing families of eclipses (solar or lunar) form threefold motifs on the Earth, due to the fact that each consecutive Saros period (223 lunations) is 6,585.321 days in length, the fractional component being about one-third of a day (or Earth rotation) out of alignment. The midpoints of the paths of totality for every third member may be joined up to reveal this threefold pattern. The result is curves called *exclegismos* (*opposite, dotted for solar and solid for lunar eclipses*).



THE DANCE OF THE MOON

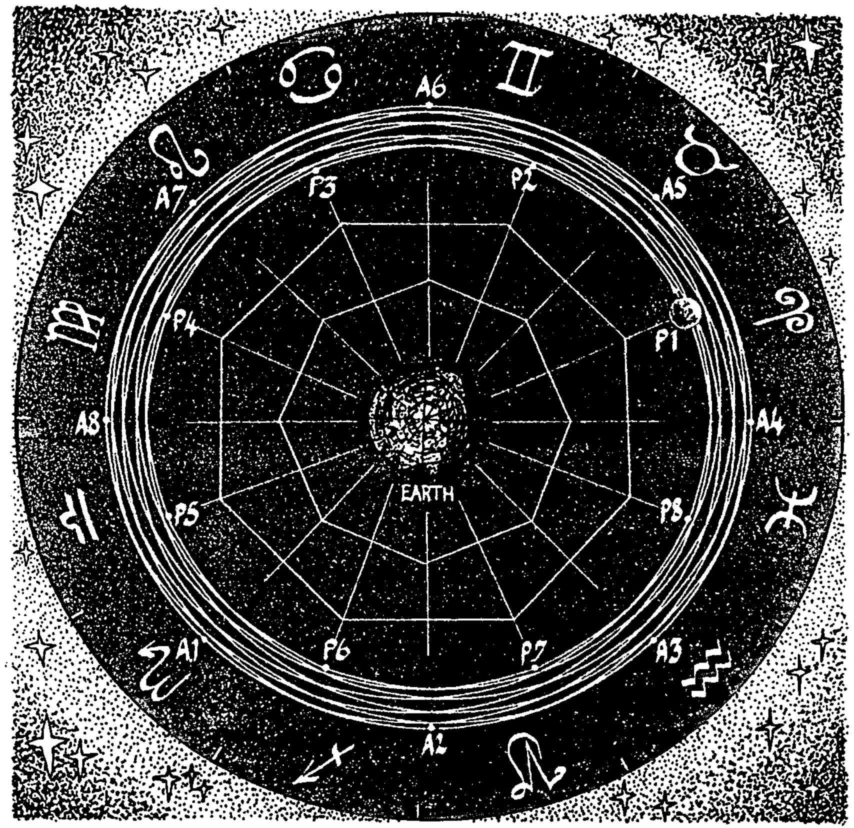
Ariadne's eight-fold web in the sky

The orbital distances of the Earth and Moon undergo periodic changes. This affects the duration and type of eclipses (*see pages 24–31*). When the Earth is nearest the Sun, strangely, in chilly January, it is said to be at *perihelion*; when farthest from the Sun it is said to be at *aphelion*.

Similarly, *perigee* occurs when the Moon is nearest the Earth, while *apogee* finds the Moon farthest from us. The line connecting these two points in the Moon's slightly elliptical orbit is called the line of *apsides*. This line or axis, the coming and going of the Moon, itself rotates, completing a cycle every 8.85 years, dividing the zodiac into eight (*opposite*) sections. A full moon at perigee appears 30 percent larger than at apogee.

The line of apsides moves counterclockwise around the zodiac by 40° 40' per year, while the nodal axis moves clockwise by 19° 20' per year (*see page 20*). The combined motion is thus 60° per year, a remarkable coincidence causing the nodes and apsides to rendezvous once more after 6 years (360°). Three of these meetings take 18 (6+6+6) years, coinciding almost exactly with the Saros cycle of 18.03 years. This is why eclipses that appear within consecutive Saros cycles are of the same type and duration.

In nautical almanacs and ephemerides, the position of the Moon is today predicted years in advance using a formula that contains over 1,500 separate factors.



A = Apogee P = Perigee

THE 19-YEAR METONIC CYCLE

the marriage of Sun and Moon

After 19 years, and within two hours of exactitude, the Sun and Moon return to the same places in the sky on the same date. This important repeat cycle is named after Meton, a fourth-century B.C. Greek astronomer. It is an astonishingly accurate repeat cycle among several other contenders (*see table opposite; the outer stone circle at Avebury once comprised 99 megaliths*).

The first-century B.C. historian Diodorus suggested that the Celts knew of the 19-year cycle. We need not be surprised at his account, for the Celts inherited the culture of the stone-circle builders, and many fine circles, particularly in southwestern Britain, such as *Boscawen-un* in Cornwall (*shown opposite*), contain 19 stones. The bluestone horseshoe at Stonehenge comprised 19 slender dressed megaliths, brought from the Preseli Mountains of west Wales, 135 miles as the crow flies. Some weighed over 4 tons!

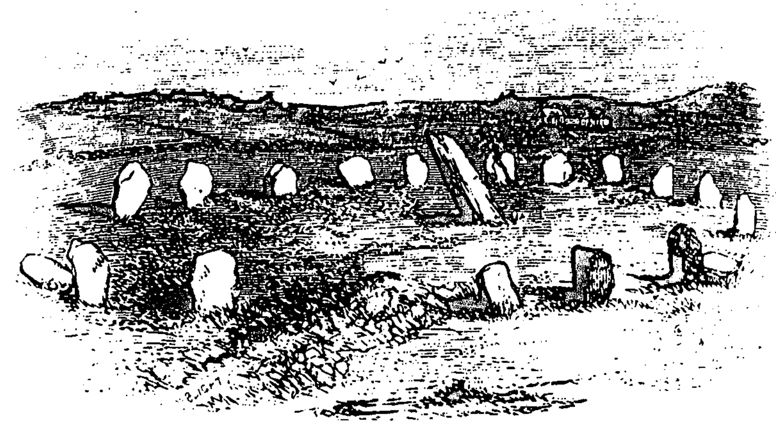
There are 12.368 lunations (full moons) in one solar year. The lunar year (12 lunations) falls short of the solar year by just under 11 days, which after 19 years accrues to 7 lunations, totaling 12 x 19 plus 7, or 235 lunations. From these numbers, the annual number of lunations may be found: 235 ÷ 19 equals $12\frac{7}{19}$, this fraction revealing the underpinning astronomy.

Nineteen solar years is 6,939.60 days; 235 lunations is 6,939.69 days. The Metonic cycle may be written as 19 x 18.618 x 19.618 days.

☽☉ CALENDAR REPEAT CYCLES ☉☽

YEARS	LUNATIONS	ERROR	EXAMPLE
3	37	3 days	
5	62	2 days	Coligny Calendar
8	99	1½ days	Avebury
19	235	2 hours	Stonehenge

☉ = 365.242 days ☽ = 29.53059 days



BOSCAWEN-UN

THE PRECESSIONAL CYCLE

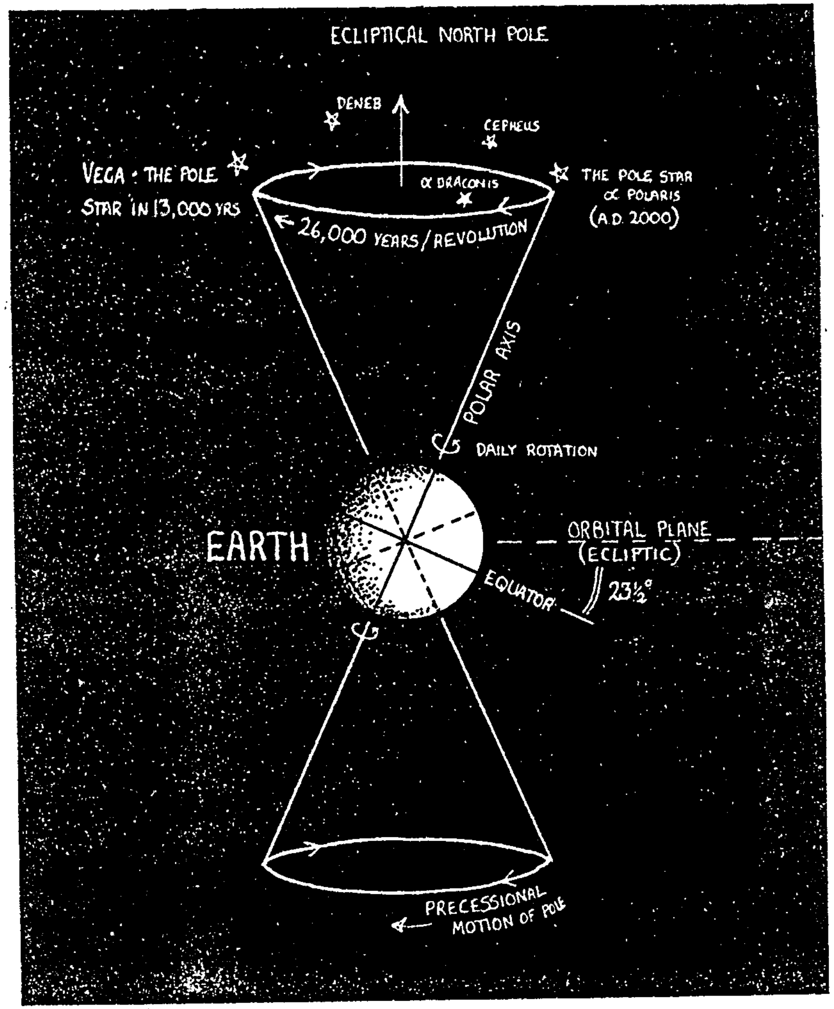
throwing a 26,000 year wobbly

The Earth's axis will not point to the present pole star forever. Like a spinning top that has tilted over a little and begun to wobble, or *precess*, the axis describes a complete circular rotation over about 25,920 years, tracing out the northern and southern circle of pole stars over this period. This is the *precessional cycle*, also known as a *Great Year* (*opposite*).

Equinoxes occur when the axial tilt of the Earth is at right angles to the Sun rather than facing toward or away from it (*see page 9*). This tilt is itself rotating backward very slowly, and the stars behind the Sun, on any given day of the year, change very slowly over time. The *Age of Pisces* commenced in A.D. 1. The *Age of Aquarius*, the next *Great Month*, will commence about A.D. 2160 when, at spring Equinox, the stars behind the Sun are those of the constellation of Aquarius. Magically, the diameter of the Moon is 2,160 miles, evoking the 2160 year length of the Great *Moonth*. Space becomes time!

In a human lifetime, precession is experienced as a single degree change in the position of the Sun against the fixed stars on a given date in the year. The precessional effect is caused by orbital asymmetries of the Sun and Moon.

The Earth's axial tilt itself varies between 21.5° and 24.8°, taking 41,000 years to complete a cycle.



THE SOLAR YEAR

packets of days, leap years, and marking time

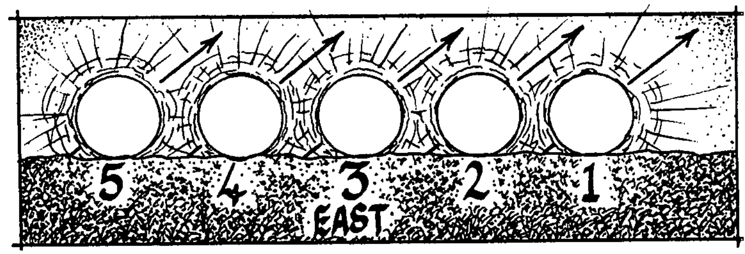
The solar year is 365.242 days long, practically 365. People who haven't ever thought the matter through will often tell you there are 365 "and a quarter" days in the year, but one cannot ever experience a quarter day in isolation. Days come in packets of one, and 365 of these make up the year, except that every fourth year an extra day slips in to make it 366.

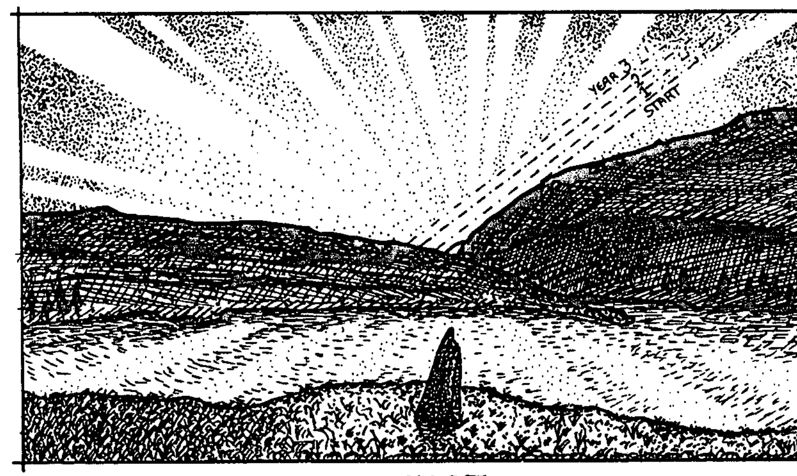
At high latitudes, consecutive sunrises around the equinoxes are spaced more than the Sun's diameter apart (*five sunrises shown opposite, top*). However, each year the vernal equinox sunrise will appear from a slightly different position on the horizon—about one quarter of a degree (*opposite, bottom*). During three years of observation, the Sun appears to rise to the left of the original alignment until, in the fourth *leap* year, it rises once more very close to the original position, the tally for the year becoming 366 days. This accounts for the "quarter day" and is the basis for the additional intercalary day, February 29.

Over longer time periods than four years one gets the chance to obtain the length of the year with even more accuracy, by observing certain key years when the Sun rises precisely behind a foresight, stone marker, or notch in a distant mountain peak, a perfect repeat solar cycle.

The best of these occurs after 33 years, 12,053 days or sunrises. This is a staggeringly accurate repeat cycle and, remarkably, seems to have been known since prehistoric times.

SUNRISES





EAST

THIRTY-THREE

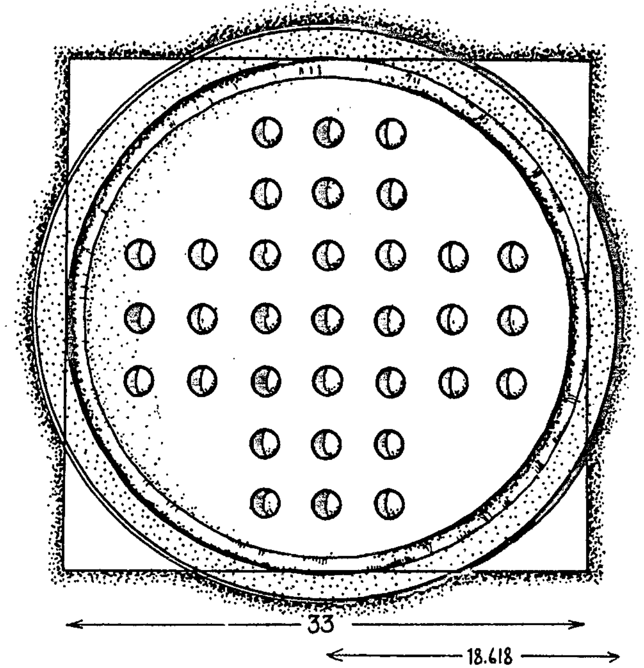
the number of the solar hero

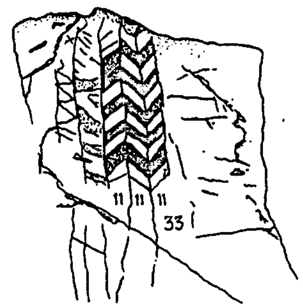
Thirty-three is a significant calendar number, which threads its secret through human culture. At a neolithic equinoctial alignment in Scotland, a cache of 33 tightly packed quartz pebbles was discovered by archaeologists. A renowned Irish megalithic site contains a stone with 33 chevrons picked out, and another has a snake motif with 33 folds (*opposite, bottom*). Ancient stories about the heroic *Tuatha de Danaan* frequently use the number 33. The first battle of *Mag Tuired* was fought by a saviour-hero *Lug* and 32 other leaders. In the second battle 33 leaders of the *Fomorc* perish, 32 plus their highest king.

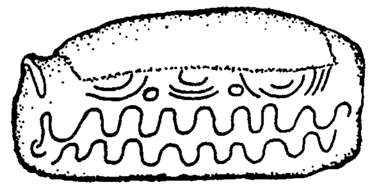
The Christian "highest king," Jesus, was crucified and resurrected at 33 years of age, rising again from behind a large stone. Islamic and Jewish calendar traditions recognize that it takes 33 solar years to complete 34 lunar years (of 12 lunations). A lunar calendar therefore cycles around the seasons, *Ramadan* falling 11 days earlier each year. The Masonic Order recognizes 33 degrees of proficiency, while the lonely game of Solitaire (*opposite, top)* consists of 33 holes and 32 balls.

These cultural artifacts share a common numerical factor, apparently derived from long-term observations of the Sun in prehistory. Pairing this cycle of the Sun with the nodal period of the Moon we may now "square the circle" by *area*, thereby solving the "third greatest problem of antiquity." Circle and square have the same area, a marriage of Heaven and Earth.

Squaring the Circle by Area







Early Irish Petroglyphs

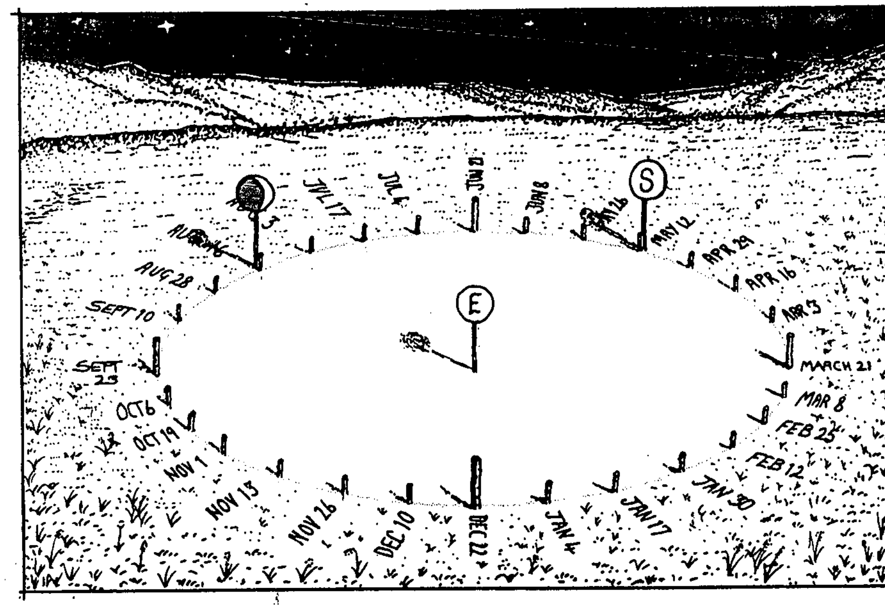
Designing a Calendar

13 months of 28 days

Timekeeping naturally starts with a calendar, which ideally should conform both to the solar year of 365 days *and* the Moon's phases. If the calendar year also divided easily into weeks and seasons and showed the phase of the Moon, the tides, and when to expect eclipses, we would be well pleased.

Starting from scratch, a calendar designer would soon discover that the Moon passes the same star in the sky every 27.3 days while the Sun takes 365 days. Dividing one by the other and choosing the nearest whole number, our designer would soon settle for 13 months of 28 days in a year. This gives a calendar year of 364 days—a number divisible by 2, 4, 7 and 13—a 52-week year with four 13-week seasons each of 91 days, a year with 13 months. All whole numbers and every year a leap year!

Practically, our designer could arrange 28 markers around the perimeter of a circle and arrange for a "moon-pole" to be moved counterclockwise once a day. A "sun-pole" would then be moved in the same direction, only thirteen times more slowly. This is the simplest way of providing a practical *soli-lunar* calendar, only requiring occasional resetting of the "Moon" (place the moon-pole opposite the sun-pole when full). If the stars were then represented on the circumference of the circle, the date, season, state of the tides (*see page 16*), lunar position, and lunar phase could be read off at a glance. We would gladly pay our designer.



The model above shows a waxing quarter moon on May 12. Twenty-eight separate and equally spaced postholes indicate the Moon's angular motion per day—about 13°. Moving a "moon-pole" counterclockwise one hole per day emulates what happens in the sky, and the Moon takes 28 days to make a circuit. A "sun-pole" is now moved one hole every 13 days, taking 364 days to make the same journey, this being the best approximation to astronomic truth with the fewest holes (97.5 percent for the Moon and 99.66 percent for the Sun). It is therefore the basis for the ancient 13 month, 364 day calendar. The improved version (page 45) also predicts eclipses and is 99.9 percent accurate for the Moon, 99.8 percent for the Sun.

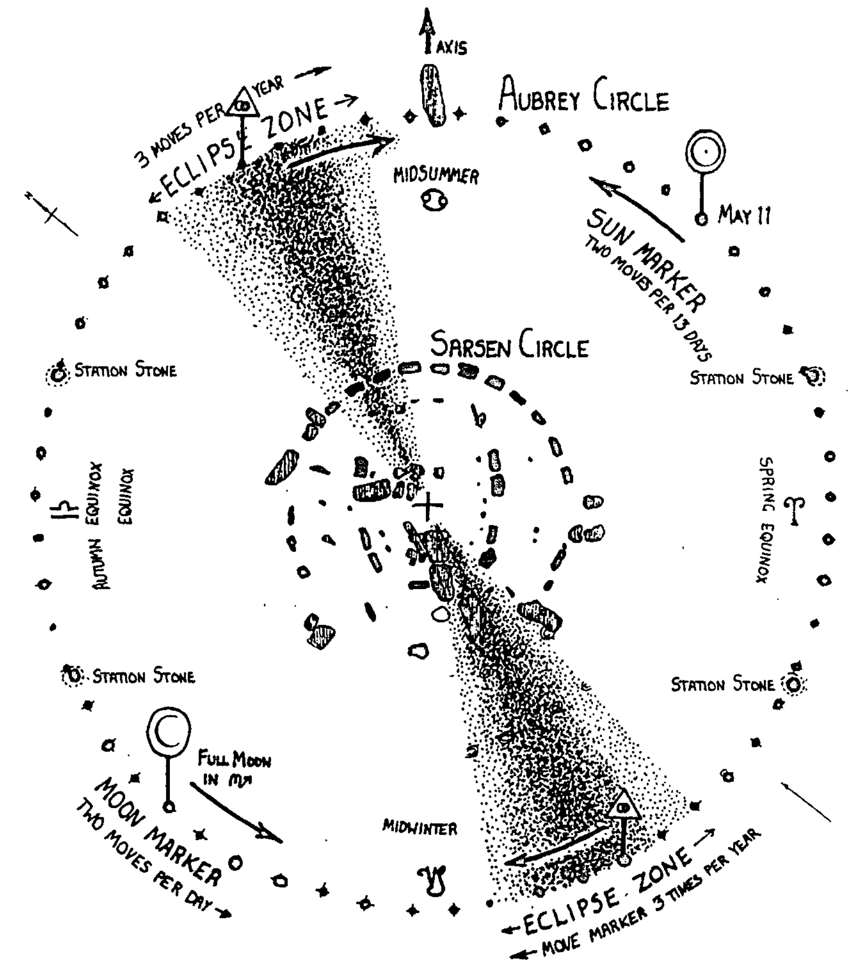
STONEHENGE

the oldest known calendar

The model shown on the previous page may be improved to include eclipse prediction. By doubling the number of postholes to 56 we exploit the useful coincidence of there being almost exactly 3 cycles of the lunar nodes in twice 28 years. The lunar nodes are here shown as two diametrically opposed triangular markers, initally placed near to the date-positions where eclipses have been recently observed.

The sun-pole now moves 2 holes counterclockwise every 13 days, and the moon-pole 2 holes counterclockwise every day. The node marker is moved three holes *clockwise* every year. When the sun-pole lies within three holes of a "node-pole," an eclipse of some sort will take place at each full or new moon, although it may not be visible at the location. If at each new moon the moon-pole is made to "skip" past the sun-pole (omitting this hole), this calendar will run for at least a year before the moon-pole needs resetting.

The design is shown opposite, hidden within the plan of Stonehenge, and indicates the date (season), Moon position, and phase, and predicts full and new moons and eclipses. The Aubrey circle (3000 B.C.) comprises 56 holes, accurately arranged round the perimeter of a 283-foot-diameter circle, predating the erection of the famous massive inner Sarsen circle. Perhaps posts were once placed in these holes; now they are filled with concrete, but this oldest known calendar would still work today.



Time and Tide

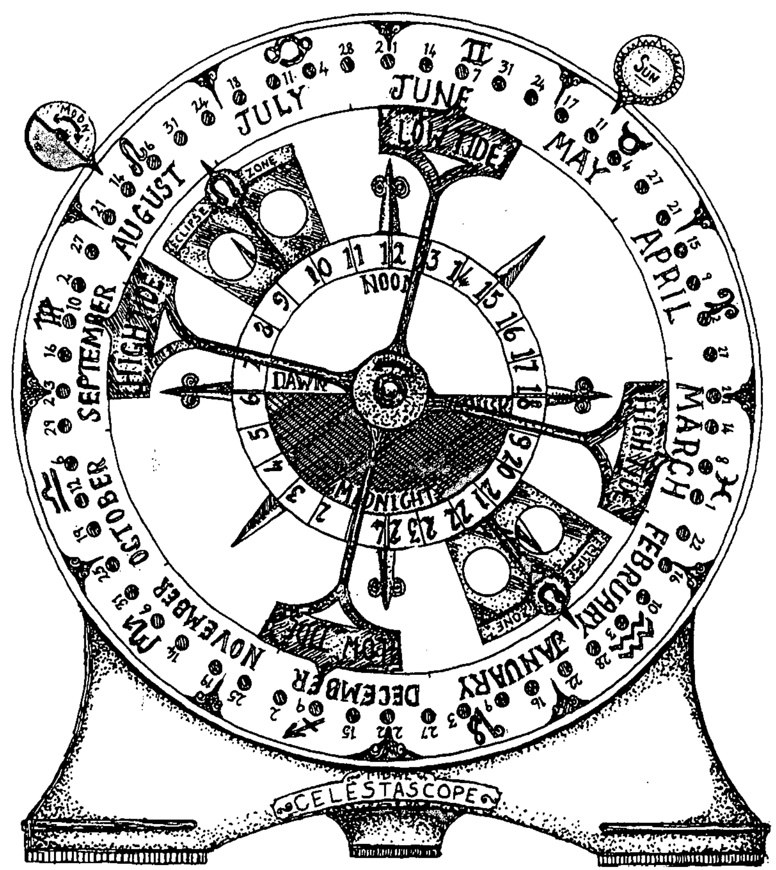
making a calendar and tidal predictor

The instrument depicted opposite is a practical calendar and eclipse and tidal predictor based on the Stonehenge Aubrey circle. It may be built and used by anyone wishing to become more aware of the rhythms of the cosmos. It is easy to learn to set up and maintain this device. You can then know the height of the tide *before* setting off for the beach!

Place the Sun marker at the current date and use an almanac to determine the Moon's position, or wait until a new or full moon. If the Moon is visible, its position can often be set approximately from its phase. The nodal axis (eclipse zone) moves clockwise three markers a year (correctly set for January 1, 2001).

The Moon's position in the sky at high tide needs first to be known for the chosen location. This varies from country to country, and to set this position, "dawn" and "dusk" are used to indicate the local horizon. The quadrant arms are then clamped at the correct angle onto the rotating central 24-hour clockface.

From now on rotate the central clock (with clamped quadrant arms) until one of the "High Tide" markers points to the current Moon position. Simply read off the time of high tide from the clock time *adjacent to the current Sun position*. For low tide, point a "Low Tide" arm and repeat the same procedure. Move the Sun and Moon every day (*as on previous page*). Big spring tides follow full and new moons, small neap tides follow the quarter moons.



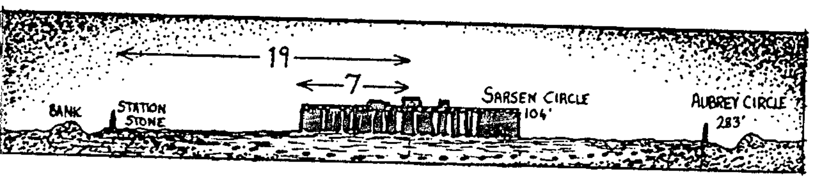
THE SILVER FRACTION

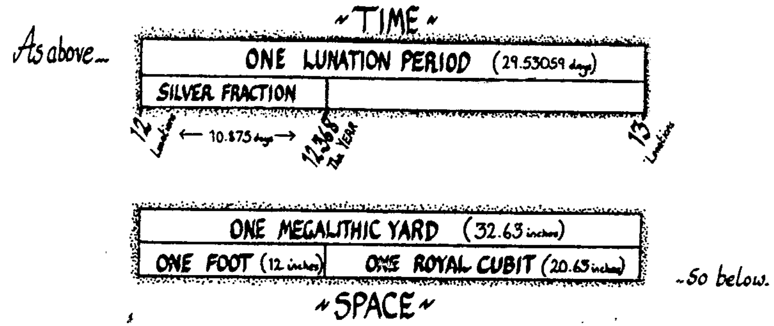
between 12 and 13 full moons

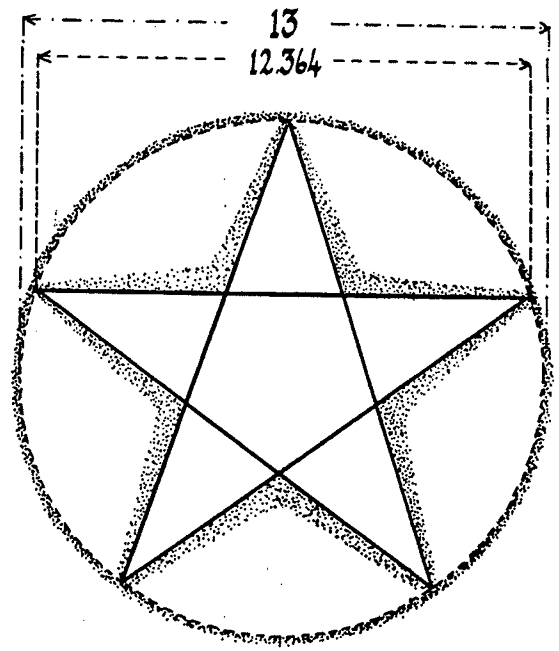
The secret of the calendar is the 11-day mismatch between the lunar year (12 lunations) and that of the solar year (12.368 lunations). Enoch called this the "over-plus of the Moon," but more poetically we may call it the *silver fraction*. Remarkably, three common units of length used by the ancient world, the *Megalithic yard*, the *Royal cubit*, and the *foot*, relate through the astronomy of the lunation and the silver fraction (*opposite, top*).

The silver fraction is actually 10.875 days in length, which is 0.368 lunations; almost exactly seven-nineteenths as a fraction. Intriguingly, the diameters of the two main circles of Stonehenge, the Aubrey circle (283 feet) and the Sarsen circle (104 feet) are in the ratio 7:19 to each other (*below*).

Simple geometry can also reveal the annual lunation figure. A pentagram drawn inside a circle of diameter 13 units has star arms of length 12.364 (*opposite, bottom*)! All 5 star arms add up to 61.82, the number of full moons in 5 years and also 100/*phi* (*to 99.9 percent—see page 28*). The famous Celtic *Coligny* calendar (100 B.C.) is based on a 5-year, 62-lunation cycle (*see page 35*). *Phi* in the sky!







THE LUNATION TRIANGLE

Solomon meets Pythagoras

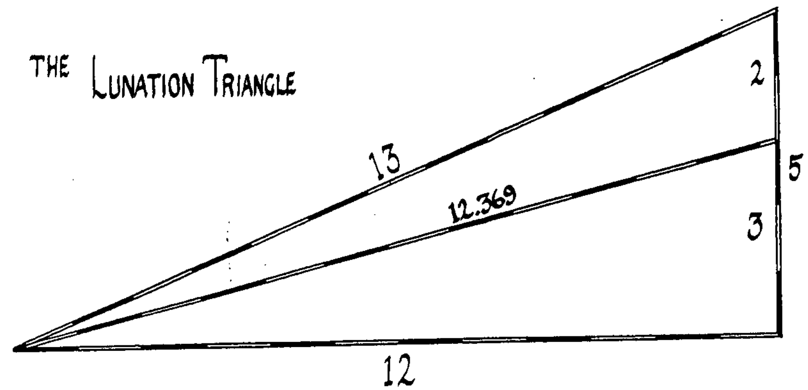
A 5:12 rectangle has a diagonal length of 13. The four station stones at Stonehenge define one (*see page 45*), as do the proportions of Solomon's Temple at Jerusalem. Twelve plus a thirteenth, as the redeeming Savior, occurs in many heroic stories, Jesus, King Arthur, and the Mayan wind god Kukulcan being examples. To the Pythagoreans the number 5 signified completeness or marriage, formed as the first male number, 3, becomes wedded to the first female number, 2.

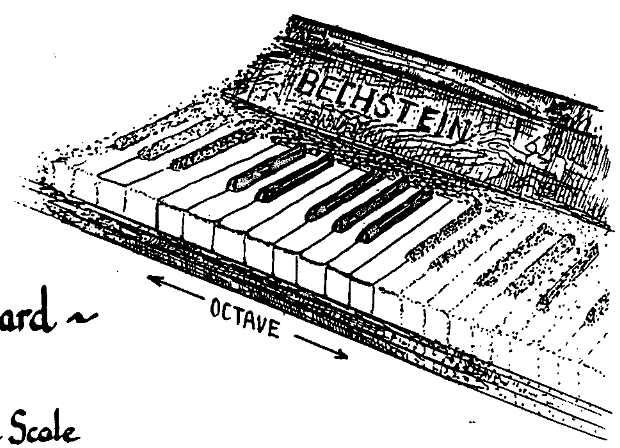
The true number of months in the year falls between 12 and 13, and in order to define a true soli-lunar calendar this figure, 12.368, must be determined. The *lunation triangle* is defined as a 5:12:13 right triangle, the second *Pythagorean triangle*, with the "5" side divided as 3:2. A new hypotenuse to this point measures 12.369. The Moon, 13, thereby becomes married to the Sun, 12, where the female, 2, joins to the male, 3. The sacred marriage of Sun and Moon, made in Heaven, is witnessed on Earth, and occurs at the musical fifth, the most harmonious interval (3:2). Musical allegories abound (*opposite, bottom*), and Solomon's throne is wisely placed at the 3:2 point in the Temple.

St John's Gospel ends with a fishy story. Jesus reappears for the third time since his resurrection and instructs his fishless disciples "cast your nets on the right side," who then catch 153 fish.

The square of the annual lunation rate is 153—12.368—to 99.99 percent.

THE LUNATION TRIANGLE





~ the Keyboard ~

The Chromatic Scale
12 notes plus a completing 13th (the octave)
5 "black" notes, arranged as 3 and 2

SUN, MOON, AND EARTH

the revealed structure of the system

The diagram opposite frames a curious symmetry. Both the eclipse year (346.62 days) and thirteen full moons (383.89 days) are almost exactly equally spaced, at 18.6 days, on either side of the solar year of 365.242 days. Because the eclipse year is the square of the lunar node period in days (18.618^2), we are now able to write that the solar year is (18.618 x 18.618) + 18.618. This is also 18.618 x 19.618. The number 18 added to 1/*phi*, *phi*, or phi^2, now delivers the following extraordinary formulas to 99.99 percent:

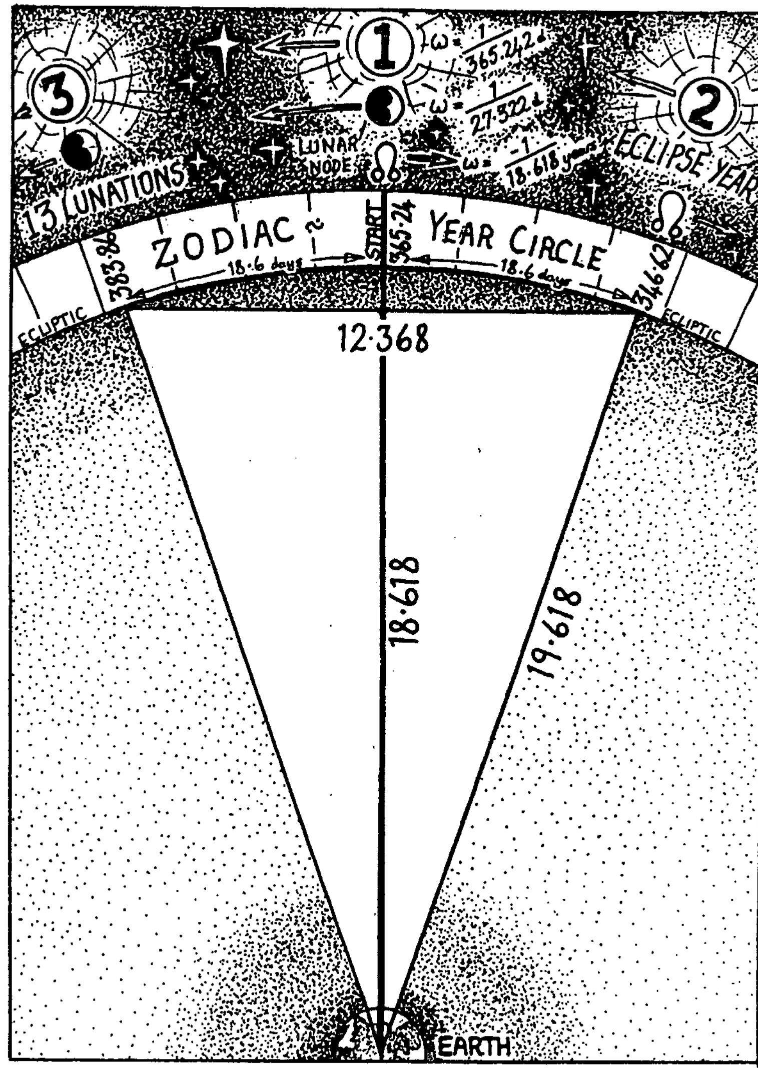
18.618 x 18.618 = 346.62 days (the eclipse year)

18.618 x 19.618 = 365.242 days (the solar year)

18.618 x 20.618 = 383.89 days (13 lunations)

The astronomy reveals the actual geometry and numerical structure of the Sun, Moon, Earth system (*opposite*). Imagine a solar eclipse at (1). The Sun then moves to meet up with the same node after an eclipse year (2), 346.62 days later. Passing the original eclipse point (1) at the end of one year, the Sun and Moon then meet for the thirteenth lunation at (3).

An isosceles triangle, drawn to fit the angles generated by the astronomy, also then defines the solar eclipse limits, and has the remarkable property of replicating the numbers shown above as ratios. In addition, the shorter side has a length 12.368, the annual lunation rate. One marvels at this revelation of cosmic order!



A Stone-Age Computer

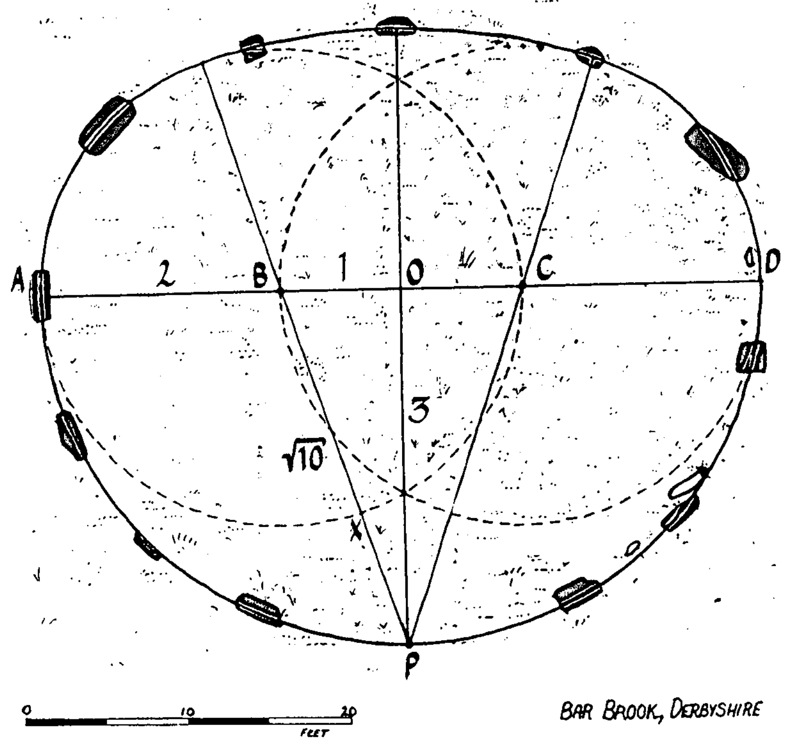
neolithic cosmology revealed

In remote places in Britain survive many examples of curiously flattened stone circles, constructed some 4,500 years ago and named *type-A* and *type-B* by Professor Thom, the discoverer of the Megalithic yard (2.72 feet, 32.64 inches, 0.83 m).

Both type-A and type-B rings invoke a Christian symbol, the *vesica piscis*, the almond shape between two overlapping circles, here applied 2,500 years before Jesus. Stones are commonly placed intelligently to the geometry (*opposite*). More astonishingly, this design invokes the same triangle we have just witnessed underpinning the structure of the Sun, Moon, Earth system!

The right triangle has side ratios $1:3:\sqrt{10}$. A rope taken from the center, O, to point P, and thence to B and A has a length $3+\sqrt{10}+2$, which is 8.16227, exactly one quarter of the Megalithic yard, in inches. If length PB is considered to represent a lunation period, then the intersection of the vesica circle cuts it at 0.368 of its length (at X). If PB is now considered to represent the solar year, then PO represents the eclipse year, this ratio being either $\sqrt{10}:3$ or 19.618 : 18.618. The ratios foot : Royal cubit : Megalithic yard may also be "read" from this exquisite device.

This most beautiful analogue of the Sun, Moon, Earth system stores their key constants *and* the ancient metrology all within itself, as ratios. An awesome glimpse of an ancient wisdom is now finally revealed.



Foot : Royal Cubit : Megalithic Yard = 1 : 1.72 : 2.72

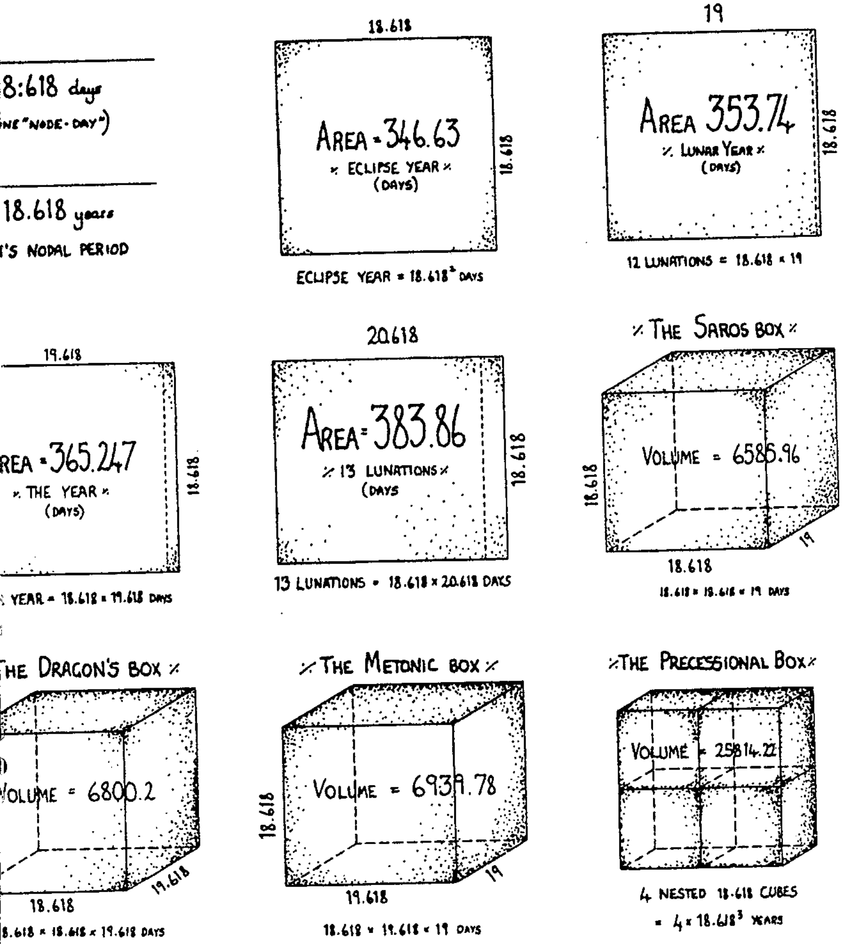
The Megalithic Yard is 2.72ft, 32.64" (4 × 8.1623")

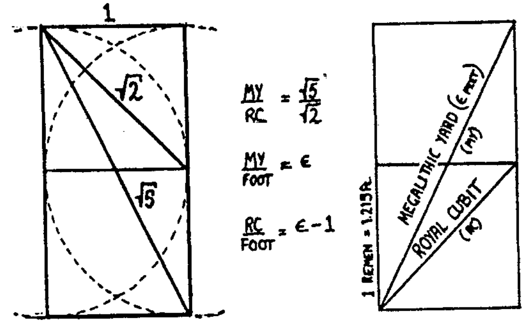
The Royal Cubit is 1.72ft, 20.64"

$\frac{\sqrt{10}}{\sqrt{10}-2} = 2.72$ $\frac{\sqrt{10}}{\sqrt{10}-5} = 1.72$ $2+3+\sqrt{10} = 8.1623$

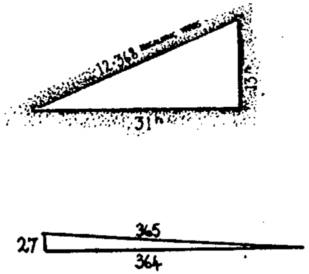
Pebbles on the Shores of Time

multidimensional solutions to local cosmology

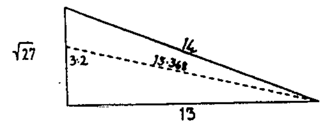




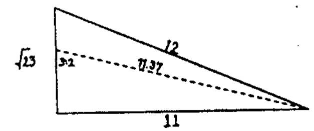
The Remen: An Egyptian unit of 1.215 ft



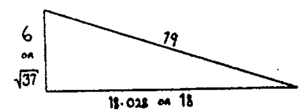
CALENDAR TRIANGLE



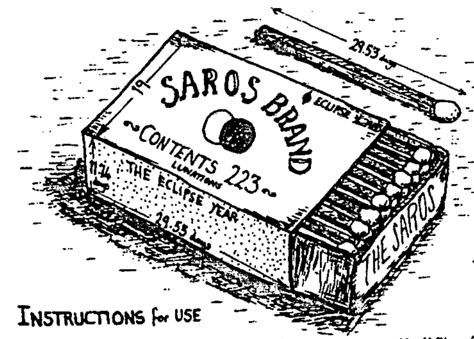
THE "SIDEREAL MONTH" TRIANGLE



THE "ECLIPSE YEAR" TRIANGLE



THE "SAROS-METONIC" TRIANGLE



INSTRUCTIONS FOR USE

1. OPEN SAROS DRAWER.
2. REMOVE A LUNATION MATCH
3. STRIKE AGAINST THE ECLIPSE YEAR AND THEN...
4. SET FIRE TO ALL PREVIOUS IDEAS YOU HAD ABOUT THE MOON.

$19 \times 11.74 = 223$

$19 \div 11.74 = \phi$

$19 \times 29.53 = \phi$ ECLIPSE YEARS

VOLUME
$= 29.53 \times 11.74 \times 19$
$= 6585.97$ days
= SAROS

TIMES

Solar Year: 365.242199 days
Lunar Year: 354.367 days
Lunation period: 29.53059 days
Eclipse Year: 346.62 days
Lunation rate: 12.36826623/year
Saros eclipse cycle: 18.03 years or 223 lunations or 6585.322 days
Lunar node cycle (Draconic year): 18.618 years or 6800.0 days
Metonic cycle: 19 years or 235 lunations or 6939.602 days
Sidereal lunar month: 27.322 days
Sun-spot cycle: 11 years
Sidereal day: 23 hours, 56 minutes, and 4 seconds
Solar tropical day (clock time): 24 hours
Lunar day (average): 24 hours, 52 minutes, and 4.31 seconds
Precessional cycle: approx 25,820 years
Tidal spacing (average): 12 hours, 26 minutes, and 2.15 seconds

LENGTHS

One foot: one degree of arc along the equator ÷ 365,242
Megalithic yard: 2.72 feet (+/- 0.003 feet).
Earth's equatorial radius: 3963.4 miles
Polar radius: 3950.0 miles
Lunar radius: 1,080 miles
Lunar distance: 222,000–253,000 miles *mean*: 240,000 miles
Sun's radius: 432,000 miles
Sun's distance (mean): 93,009,000 miles

ANGLES & RATIOS

Lunar orbit inclination to Earth-Sun plane: 5° 8' 30"
Solar angular diameter (mean): 0° 32'
Lunar angular diameter (mean): 0° 31' 30"
Earth's axial tilt: 23° 27'
Earth to Moon density ratio: 1.6 : 1
Earth to Moon mass ratio: 81:1
Eccentricity of lunar orbit: 1/18

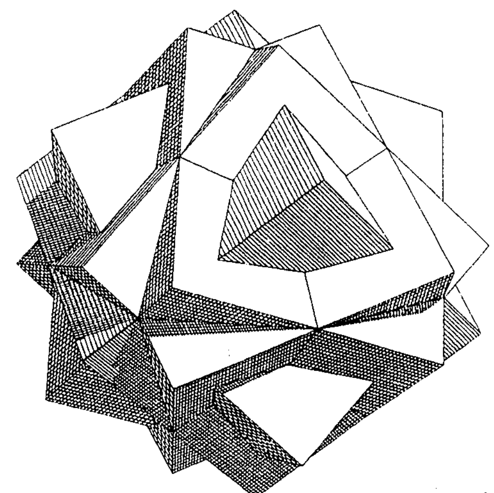
SOME DEFINITIONS

Lunation period: Time between consecutive new moons (or full moons).
Ecliptic: The Sun's apparent path through the zodiacal belt of stars, seen from Earth.
Solar tropical year: Time between consecutive sping equinoxes.
Precessional (Great) year: Time for the zodiac to rotate (backward) around the calendar.
Syzygy: Sun, Moon, and Earth in a line.
Solar/Lunar day: Time between consecutive south transits of the Sun/Moon.
Lunar year: 12 lunations.
Sidereal lunar period: Time for the Moon to return to the same longitude (or star).
Lunar nodes: Two opposite points where the Moon's path crosses the ecliptic.
Eclipse year: Time between consecutive conjunctions of the Sun and north node.

IRRATIONALS

e = 2.718282 . . .   √2 = 1.414214 . . .   *phi*(ø) = 1.618034 . . .

PLATONIC & ARCHIMEDEAN SOLIDS



Originally published in Wales by Wooden Books Ltd. in 1998; first published in the United States of America in 2002 by Walker Publishing Company, Inc.

Published simultaneously in Canada by Fitzhenry and Whiteside, Markham, Ontario L3R 4T8

Printed on recycled paper.

For information about permission to reproduce selections from this book, write to Permissions, Walker & Company, 435 Hudson Street, New York, New York 10014

Library of Congress Cataloging-in-Publication Data

Sutton, Daud.
Platonic & Archimedean solids/written and illustrated by Daud Sutton.
p.cm.
Originally published: Presteigne, Powys, Wales : Wooden Books, 1998.
ISBN 0-8027-1386-6 (alk. paper)
1. Polyhedra. I. Title: Platonic and Archimedean solids. II. Title.

QA491 .S88 2002
516'.15--dc21

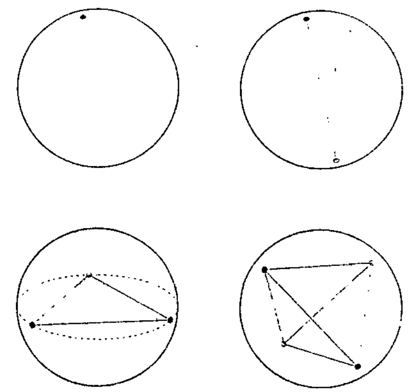
2001055924

Visit Walker & Company's Web site at www.walkerbooks.com

Printed in the United States of America

2 4 6 8 10 9 7 5 3 1

PLATONIC
& ARCHIMEDEAN
SOLIDS



written and illustrated by

Daud Sutton

WOODEN BOOKS

Walker & Company
New York

بسم الله الرحمن الرحيم

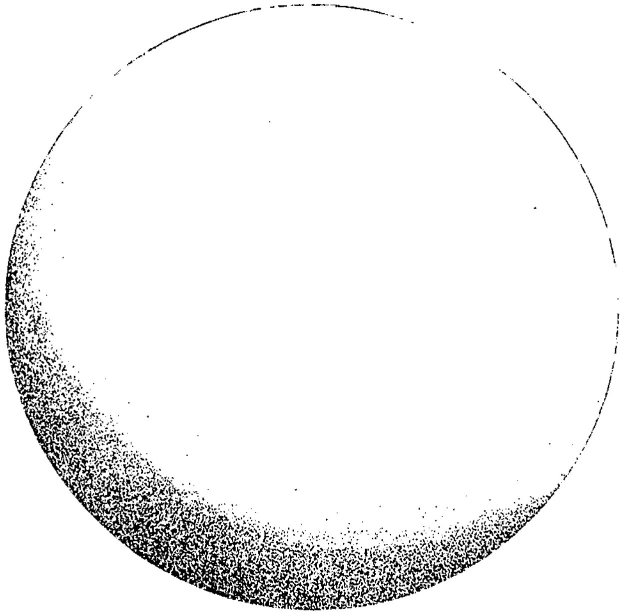
In The Name of God,
The Compassionate, The Merciful

This book is dedicated to Professor Keith Critchlow, whose teaching made it possible.

I am indebted to the many geometers, authors, and artists who have explored the world of polyhedra.

Thanks to my family and friends for comments and contributions.

Contents



INTRODUCTION

Imagine a sphere.

It is unity's perfect symbol. Each point on its surface is identical to every other, equidistant from the unique point at its center.

Establishing a single point on the sphere allows others to be defined in relation to it. The simplest and most obvious relationship is with the point directly opposite, found by extending a line through the sphere's center to the other side. Add a third point and space all three as far from each other as possible to define an equilateral triangle. The three points lie on a circle with a radius equal to the sphere's and sharing its center, an example of the largest circles possible on a sphere, known as great circles. Point, line, and triangle occupy zero, one, and two dimensions respectively. It takes a minimum of four points to define an uncurved three-dimensional form.

This small book charts the unfolding of number in three-dimensional space through the most fundamental forms derived from the sphere. A cornerstone of mathematical and artistic inquiry since antiquity, after countless generations these beautiful forms continue to intrigue and inspire.

Cairo, Summer 2001

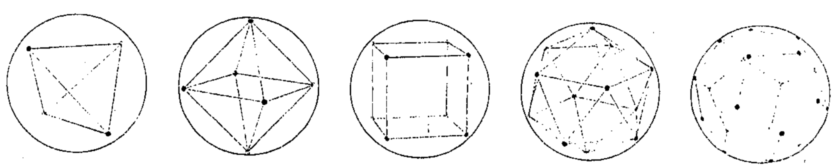
THE PLATONIC SOLIDS

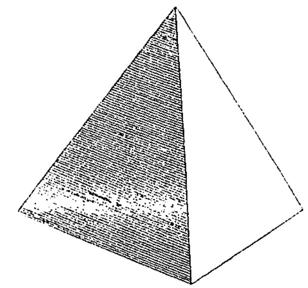
beautiful forms unfold from unity

Imagine you are on a desert island; there are sticks and sheets of bark. If you start experimenting with making three-dimensional structures you may well discover five "perfect" shapes. In each case they look the same from any *vertex* (corner point), their faces are all made of the same regular shape, and every edge is identical. Their vertices are the most symmetrical distributions of four, six, eight, twelve, and twenty points on a sphere (*below*).

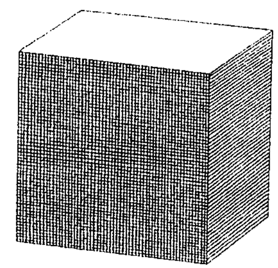
These forms are examples of *polyhedra*, literally "many seats," and, as the earliest surviving description of them as a group is in Plato's *Timaeus*, they are often called the Platonic solids. Plato lived from 427 B.C. to 347 B.C., but there is evidence that they were discovered much earlier (*see page 20*).

The *cube*, with its six square faces, is well known. The other four have names deriving from their numbers of faces. Three of the solids have faces of equilateral triangles: the *tetrahedron* is made from four, the *octahedron* eight, and the *icosahedron* twenty. The *dodecahedron* has twelve regular pentagonal faces. The following ten pages will describe these striking three-dimensional forms in greater detail.

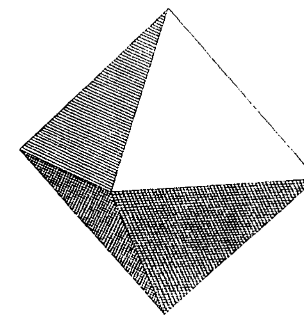




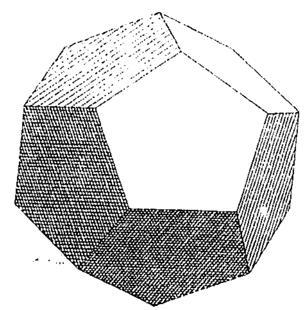
tetrahedron



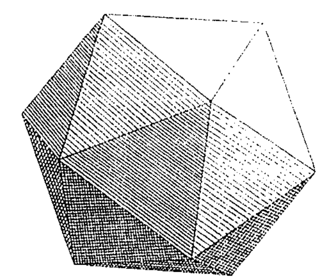
cube



octahedron



dodecahedron



icosahedron

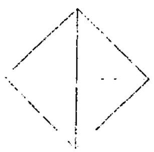
THE TETRAHEDRON

4 faces, 6 edges, 4 vertices

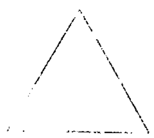
The tetrahedron is composed of four equilateral triangles, with three meeting at every vertex. Its vertices can also be defined by the centers of four touching spheres (*opposite, bottom right*). Plato associated its form with the element of fire because of the penetrating acuteness of its edges and vertices, and because it is the simplest and most fundamental of the regular solids. The Greeks also knew the tetrahedron as *puramis*, from which the word *pyramid* is derived. Curiously the Greek word for fire is *pur*.

The tetrahedron has three 2-fold axes of symmetry, passing through the midpoints of its edges, and four 3-fold axes, each passing through one vertex and the opposite face center (*below*). Any polyhedron with these rotation axes has *tetrahedral symmetry*.

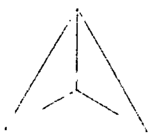
Each Platonic solid is contained by its *circumsphere*, which just touches every vertex. The solids also define two more spheres: their *midsphere*, which passes through the midpoint of every edge, and their *insphere*, which is contained by the solid, perfectly touching the center of every face. For the tetrahedron the *inradius* is one-third of the *circumradius* (*opposite, bottom left*).



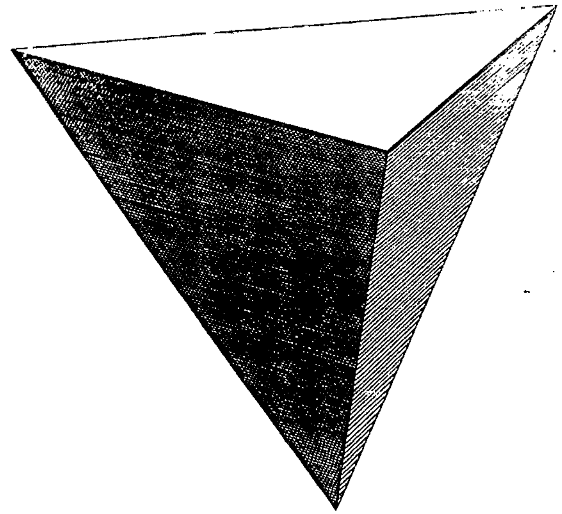
edge on : 2-fold

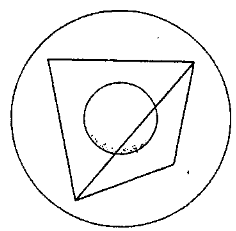


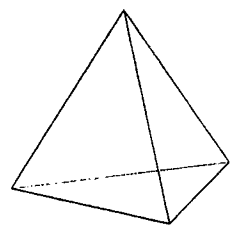
face on : 3-fold

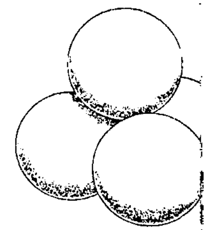


from vertex : 3-fold









The Octahedron

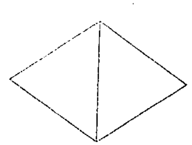
8 faces, 12 edges, 6 vertices

The octahedron is made of eight equilateral triangles, four meeting at every vertex. Plato considered the octahedron an intermediary between the tetrahedron, or fire, and the icosahedron, or water and thus ascribed it to the element of air. The octahedron has six 2-fold axes passing through opposite edges, four 3-fold axes passing through its face centers, and three 4-fold axes passing through opposite vertices (*below*). Any polyhedron combining these rotation axes is said to have *octahedral symmetry*.

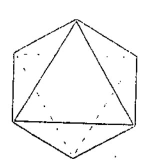
Greek writings attribute the discovery of the octahedron and icosahedron to Theaetetus of Athens (417 B.C.–369 B.C.). Book XIII of Euclid's *Elements* (*see page 14*) is thought to be based on Theaetetus' work on the regular solids.

The octahedron's circumradius is bigger than its inradius by a factor of $\sqrt{3}$ (*see page 55*). The same relationship occurs between the circumradius and inradius of the cube, and between the circumradius and midradius of the tetrahedron.

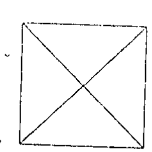
The tetrahedron, the octahedron and the cube are all found in the mineral kingdom. Mineral diamonds often form octahedra.



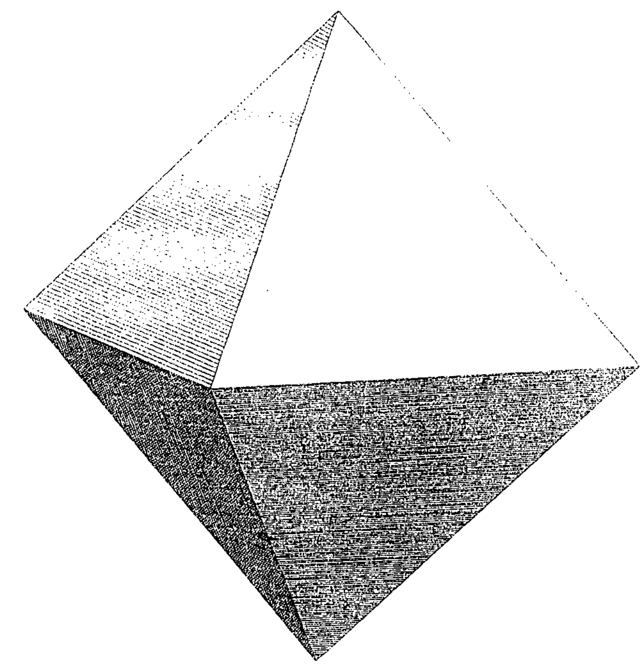
edge on : 2-fold

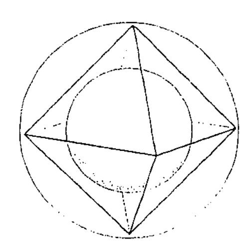


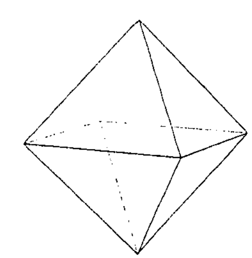
face on : 3-fold

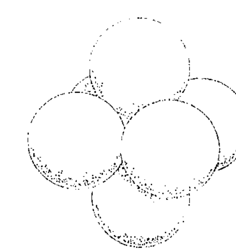


from vertex : 4-fold









The Icosahedron

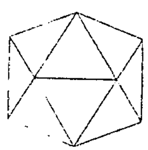
20 faces, 30 edges, 12 vertices

The icosahedron is composed of twenty equilateral triangles, five to a vertex. It has fifteen 2-fold axes, twenty 3-fold axes, and twelve 5-fold axes (*below*), known as *icosahedral symmetry*. When the tetrahedron, octahedron, and icosahedron are made of identical triangles, the icosahedron is the largest. This led Plato to associate the icosahedron with water, the densest and least penetrating of the three fluid elements—fire, air, and water.

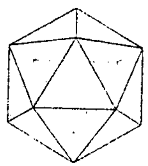
The angle where two faces of a polyhedron meet at an edge is known as a *dihedral angle*. The icosahedron is the Platonic solid with the largest dihedral angles.

If you join the two ends of an icosahedron's edge to the center of the solid an isosceles triangle is defined. This triangle is the same as those that make up the faces of the Great Pyramid at Giza in Egypt.

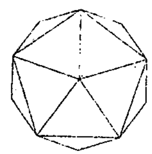
Arranging twelve equal spheres to define an icosahedron leaves space at the center for another sphere just over nine-tenths as wide as the others (*opposite, lower right*).



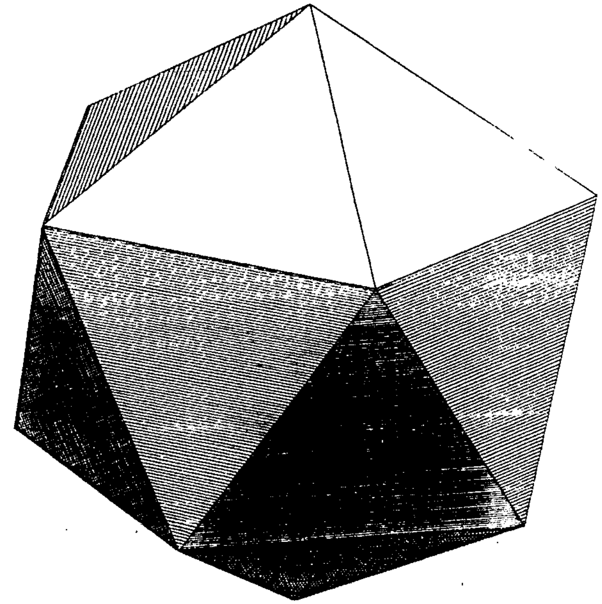
edge on : 2-fold

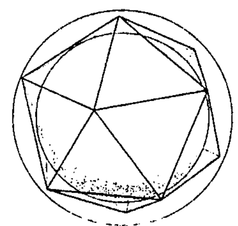


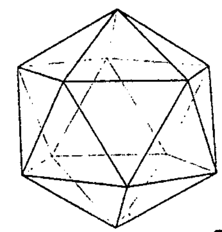
face on : 3-fold

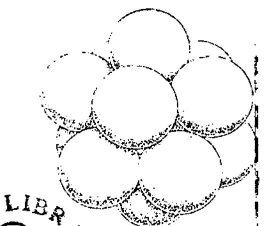


from vertex : 5-fold









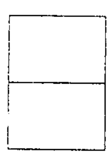
THE CUBE

6 faces, 12 edges, 8 vertices

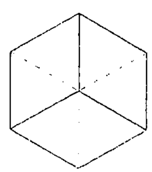
The cube has octahedral symmetry (*below*). Plato assigned it to the element of earth due to the stability of its square bases. Aligned to our experience of space it faces forward, backward, right, left, up, and down, corresponding to the six directions north, south, east, west, zenith, and nadir. Six is the first *perfect number*, with factors adding up to itself $(1+2+3=6)$.

Add the cube's twelve edges, the twelve face diagonals, and the four interior diagonals to find a total of twenty-eight straight paths joining the cube's eight vertices to each other. Twenty-eight is the second perfect number $(1+2+4+7+14=28)$.

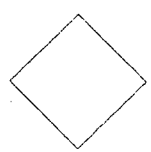
Islam's annual pilgrimage is to the Kaaba, literally cube, in Mecca. The sanctuary of the Temple of Solomon was a cube, as is the crystalline New Jerusalem in Saint John's revelation. In 430 B.C. the oracle at Delphi instructed the Athenians to double the volume of the cubic altar of Apollo while maintaining its shape. "Doubling the cube," as the problem became known, ultimately proved impossible using Euclidean geometry alone.



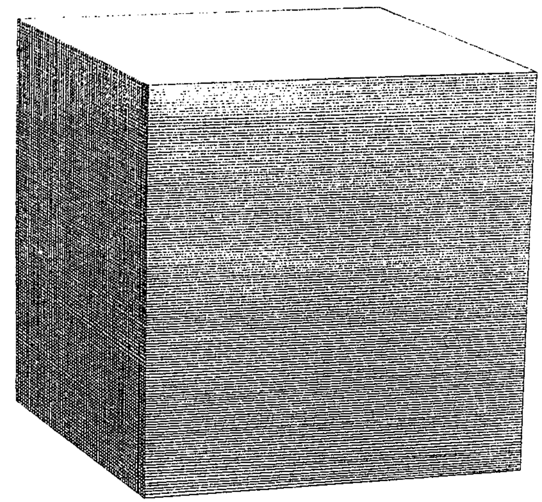
edge on : 2-fold

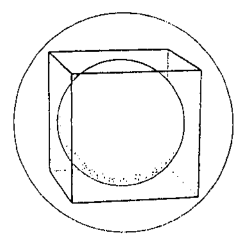


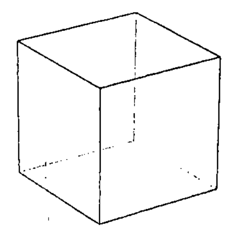
from vertex : 3-fold

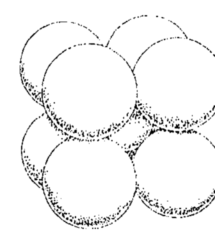


face on : 4-fold









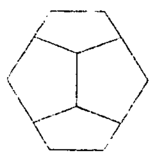
The Dodecahedron

12 faces, 30 edges, 20 vertices

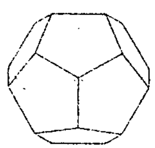
The beautiful dodecahedron has twelve regular pentagonal faces, three of which meet at every vertex. Its symmetry is icosahedral (*below*). Like the tetrahedron, or pyramid, and the cube, the dodecahedron was known to the early Pythagoreans and was commonly referred to as *the sphere of twelve pentagons*. Having detailed the other four solids and ascribed them to the elements, Plato's *Timaeus* says enigmatically, "There remained a fifth construction which God used for embroidering the constellations on the whole heaven."

A dodecahedron sitting on a horizontal surface has vertices lying in four horizontal planes that cut the dodecahedron into three parts. Surprisingly, the middle part is equal in volume to the others, so each is one-third of the total! Also, when set in the same sphere, the surface areas of the icosahedron and dodecahedron are in the same ratio as their volumes.

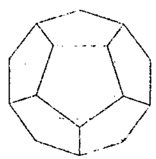
"Fool's Gold," or iron pyrite, forms crystals much like the dodecahedron, but don't be fooled, their pentagonal faces are not regular and their symmetry is tetrahedral.



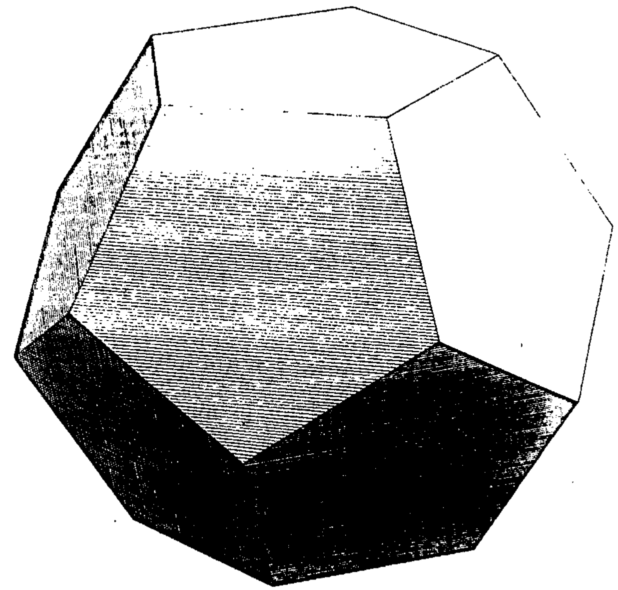
edge on : 2-fold

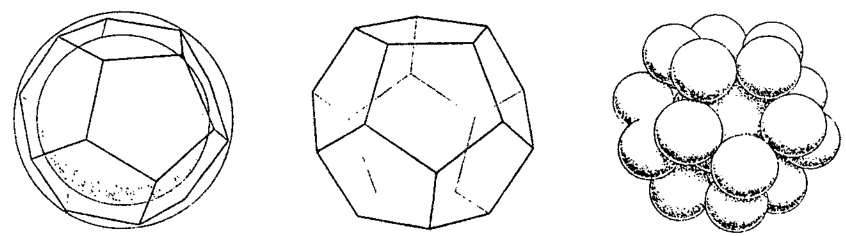


from vertex : 3-fold



face on : 5-fold





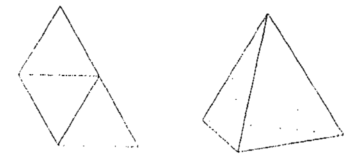
A Short Proof

are there really only five?

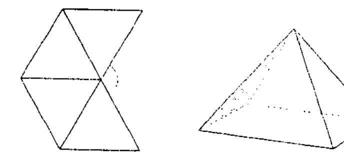
A regular polygon has equal sides and angles. A regular polyhedron has equal regular polygon faces and identical vertices. The five Platonic solids are the only convex regular polyhedra.

At least three polygons are needed to make a *solid angle*. Using equilateral triangles this is possible with three (*A*), four (*B*), and five (*C*) around a point. With six the result lies flat (*D*). Three squares make a solid angle (*E*), but with four (*F*) a limit similar to six triangles is reached. Three regular pentagons form a solid angle (*G*), but there is no room, even lying flat, for four or more. Three regular hexagons meeting at a point lie flat (*H*), and higher polygons cannot meet with three around a point, so a final limit is reached. Since only five solid angles made of identical regular polygons are possible, there are at most five possible convex regular polyhedra. Incredibly, all five regular solid angles repeat to form the regular polyhedra. This proof is given by Euclid of Alexandria (c. 325 B.C.–265 B.C.) in Book XIII of his *Elements*.

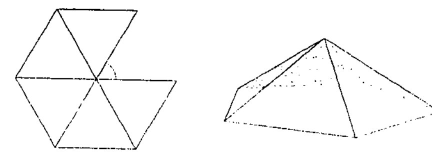
The angle left as a gap when a polyhedron's vertex is folded flat is its *angle deficiency*. René Descartes (1596–1650) discovered that the sum of a convex polyhedron's angle deficiencies always equals 720°, or two full turns. Later, in the eighteenth century, Leonhard Euler (1707–1783) noticed another peculiar fact: In every convex polyhedron the number of faces minus the number of edges plus the number of vertices equals two.



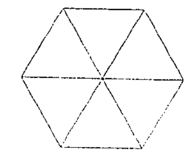
A



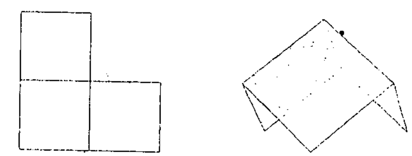
B



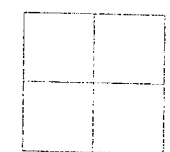
C



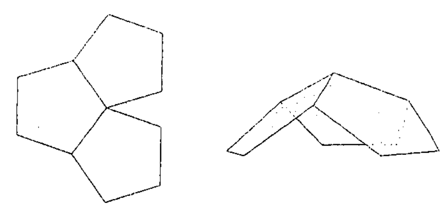
D



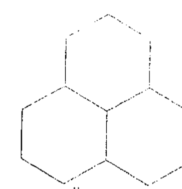
E



F



G



H

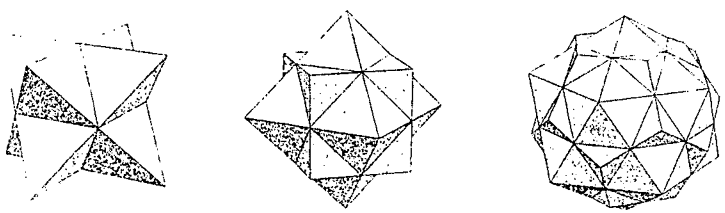
All Things in Pairs

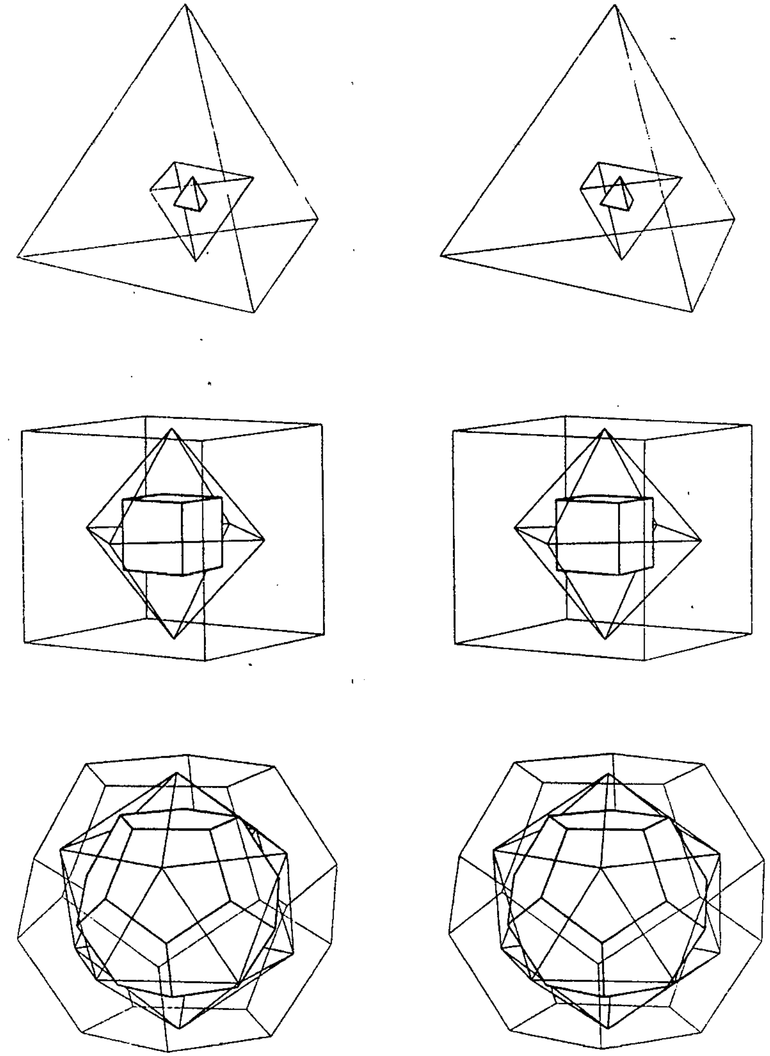
Platonic solids two by two

What happens if we join the face centers of the Platonic solids? Starting with a tetrahedron, we discover another, inverted, tetrahedron. The faces of a cube produce an octahedron, and an octahedron creates a cube. The icosahedron and dodecahedron likewise produce each other. Two polyhedra whose faces and vertices correspond perfectly are known as each other's *duals*. The tetrahedron is *self-dual*. Dual polyhedra have the same number of edges and the same symmetries.

The illustrations opposite are stereogram pairs. Hold the book at arms length and place a finger vertically, midway to the page. Focus on the finger and then bring the central blurred image into focus. The image should jump into three dimensions.

Dual pairs of Platonic solids can be joined with their edges touching at their midpoints to give the compound polyhedra shown below. Everything in creation has its counterpart or opposite, and the dual relationships of the Platonic solids are a beautiful example of this principle.





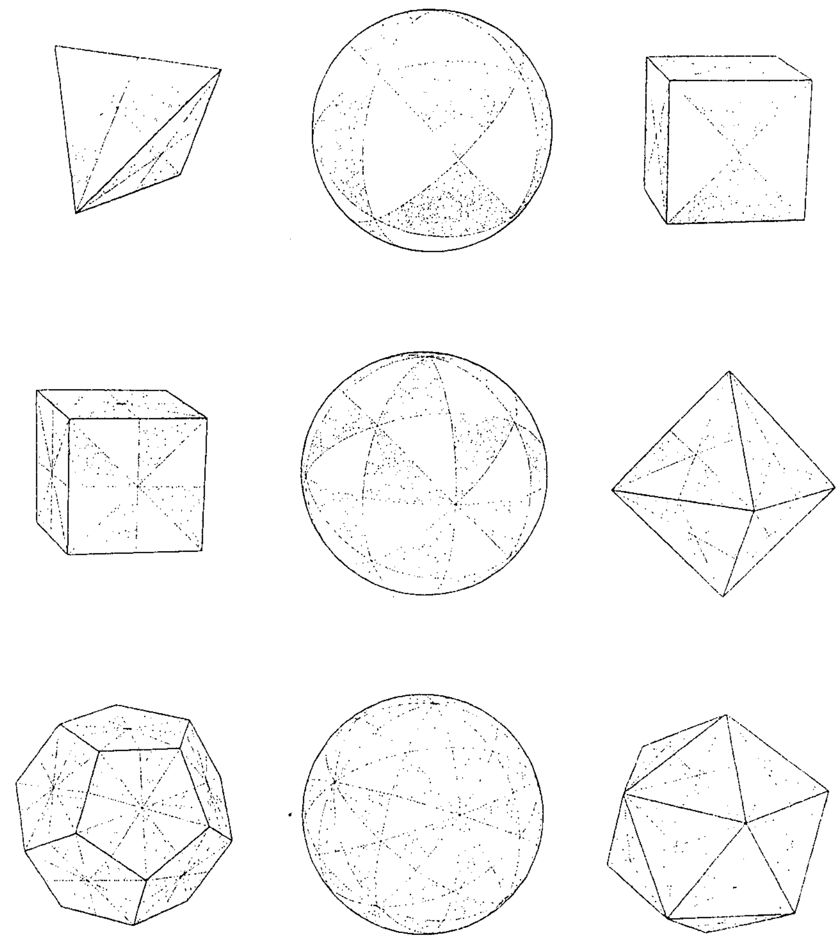
AROUND THE GLOBE

in elegant ways

Plato's cosmology constructs the elemental solids from two types of right-triangular atoms. The first atom is half an equilateral triangle, six of which then compound to produce larger equilateral triangles; these go on to form the tetrahedron, octahedron, and icosahedron. The second triangular atom is a diagonally halved square, which appears in fours, making squares that then form cubes.

The Platonic solids have planes of symmetry dividing them into mirror-image halves. The tetrahedron has six, the octahedron and cube have nine, and the icosahedron and dodecahedron have fifteen. When the tetrahedron, octahedron, and icosahedron are constructed from Plato's triangular atoms, paths are defined that make their mirror planes explicit. The cube, however, needs twice as many triangular divisions as Plato gave it (*top row*) to delineate all its mirror planes (*middle row*).

Projecting the subdivided Platonic solids onto their circumspheres produces three spherical systems of symmetry. Each spherical system is defined by a characteristic spherical triangle with one right angle and one angle of one-third of a half turn. Their third angles are respectively one-third of a half turn (*top row*), one-quarter of a half turn (*middle row*), and one-fifth of a half turn (*bottom row*). This sequence of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ elegantly inverts the Pythagorean whole number triple 3, 4, 5.



Round and Round

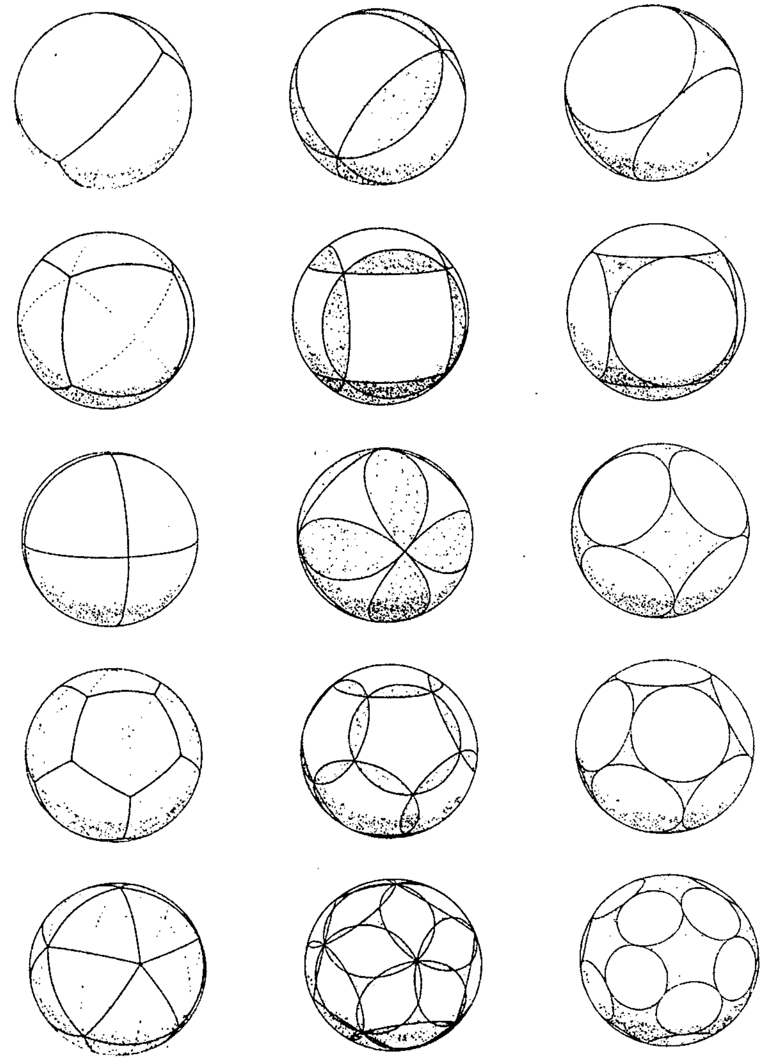
lesser circles

Any navigator will tell you that the shortest distance between two points on a sphere's surface is always an arc of a *great circle*. When a polyhedron's edges are projected onto its circumsphere the result is a set of great circle arcs known as a *radial projection*. The left-hand column opposite shows the radial projections of the Platonic solids with their great circles shown as dotted lines.

A spherical circle smaller than a great circle is called a *lesser circle*. Tracing a circle around all the faces of the Platonic solids set in their circumspheres generates the patterns of lesser circles, shown in the middle column. Book XIV of Euclid's *Elements* proves that when set in the same sphere, the lesser circles around the dodecahedron's faces (*fourth row*) are equal to the lesser circles around the icosahedron's faces (*fifth row*). The same is true of the cube (*second row*) and the octahedron (*third row*) as a pair.

Shrink the lesser circles in the middle column until they just touch each other to define the five spherical curiosities in the right-hand column. Many neolithic carved stone spheres have been found in Scotland with the same patterns as the first four of these arrangements. The dodecahedral carvings of twelve circles on a sphere, some 4,000 years old, are the earliest known examples of manmade designs with icosahedral symmetry.

Large lesser circle models can be made from circles of willow, or cheap hula-hoops, lashed together with wire, string, or tape.

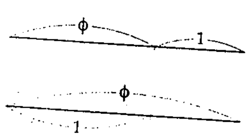


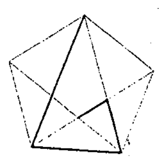
The Golden Ratio

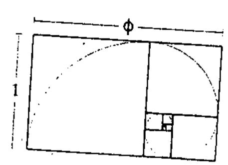
and some intriguing juxtapositions

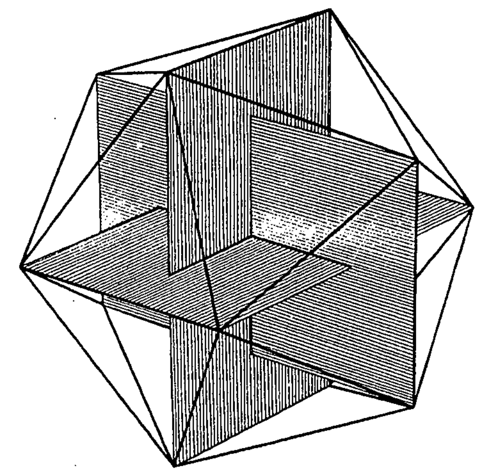
Dividing a line so that the shorter section is to the longer as the longer section is to the whole line defines the *golden ratio* (*below*). It is an irrational number, inexpressible as a simple fraction (*see page 55*). Its value is one plus the square root of five, divided by two—approximately 1.618. It is represented by the Greek letter ϕ (*phi*) or sometimes by τ (*tau*). ϕ has intimate connections with unity; ϕ times itself (ϕ^2) is equal to ϕ plus one (2.618 . . .), and one divided by ϕ equals ϕ minus one (0.618 . . .). It is innately related to five-fold symmetry; each successive pair of heavy lines in the pentagram below is in the golden ratio.

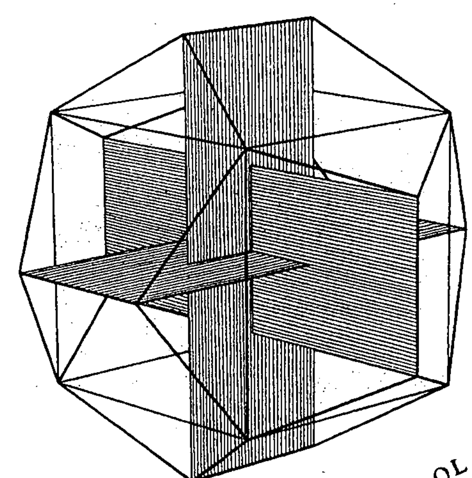
A *golden rectangle* has sides in the golden ratio. If a square is removed from one side, the remaining rectangle is another golden rectangle. This process can continue indefinitely and establishes a golden spiral (*below right*). Remarkably, an icosahedron's twelve vertices are defined by three perpendicular golden rectangles (*opposite, top*). The dodecahedron is even richer. Twelve of its twenty vertices are defined by three perpendicular ϕ^2 rectangles, and the remaining eight vertices are found by adding a cube of edge length ϕ (*opposite, bottom*).











Polyhedra Within Polyhedra

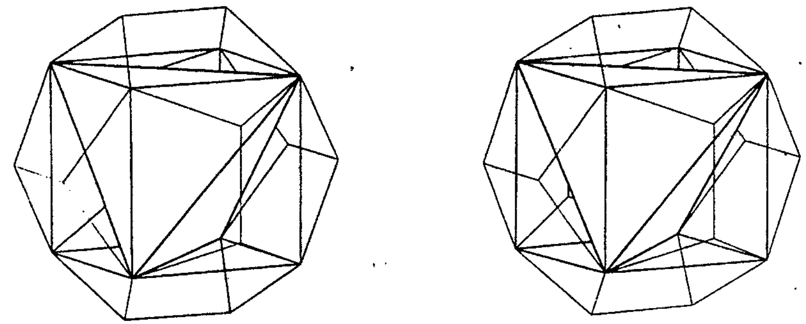
and so proceed ad infinitum

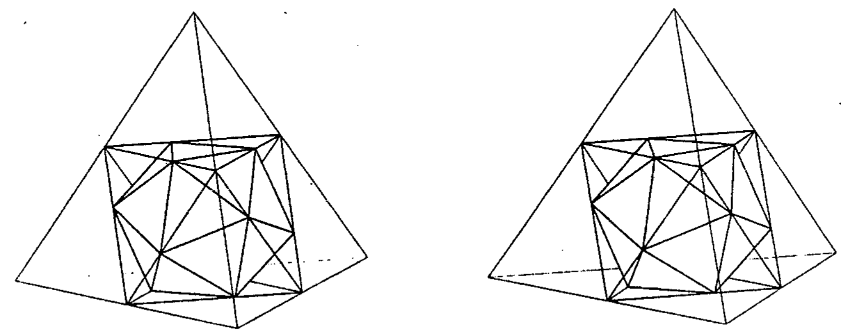
The Platonic solids fit together in remarkable and fascinating ways. Page 54 shows many of those relationships. The upper stereogram pair opposite shows a dodecahedron with edge length one. Nested inside it is a cube, edge length ϕ, and a tetrahedron, edge length $\sqrt{2}$ (*see page 55*) times the cube's. The tetrahedron occupies one-third of the cube's volume.

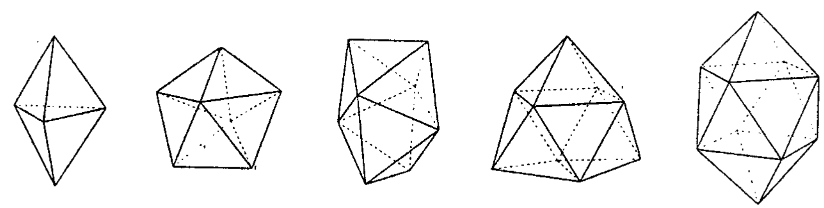
In the lower stereogram pair opposite, the six edge midpoints of the tetrahedron define the six vertices of an octahedron. As well as halving the tetrahedron's edges this octahedron has half its surface area and half its volume, perfectly embodying the musical octave ratio of 1:2. Similarly the twelve edges of the octahedron correspond to the twelve vertices of a nested icosahedron. The icosahedron's vertices cut the octahedron's edges perfectly into the golden ratio.

Imagine these two sets of nestings combined to give all five Platonic solids in one elegant arrangement. Since the outer dodecahedron defines a larger icosahedron by their dual relationship, and the inner icosahedron likewise defines a smaller dodecahedron, the nestings can be continued outward and inward to infinity.

The tetrahedron, octahedron, and icosahedron, made entirely from equilateral triangles, are known as *convex deltahedra*, after the Greek letter Δ (*delta*). The five other possible convex deltahedra are shown in the bottom row opposite.





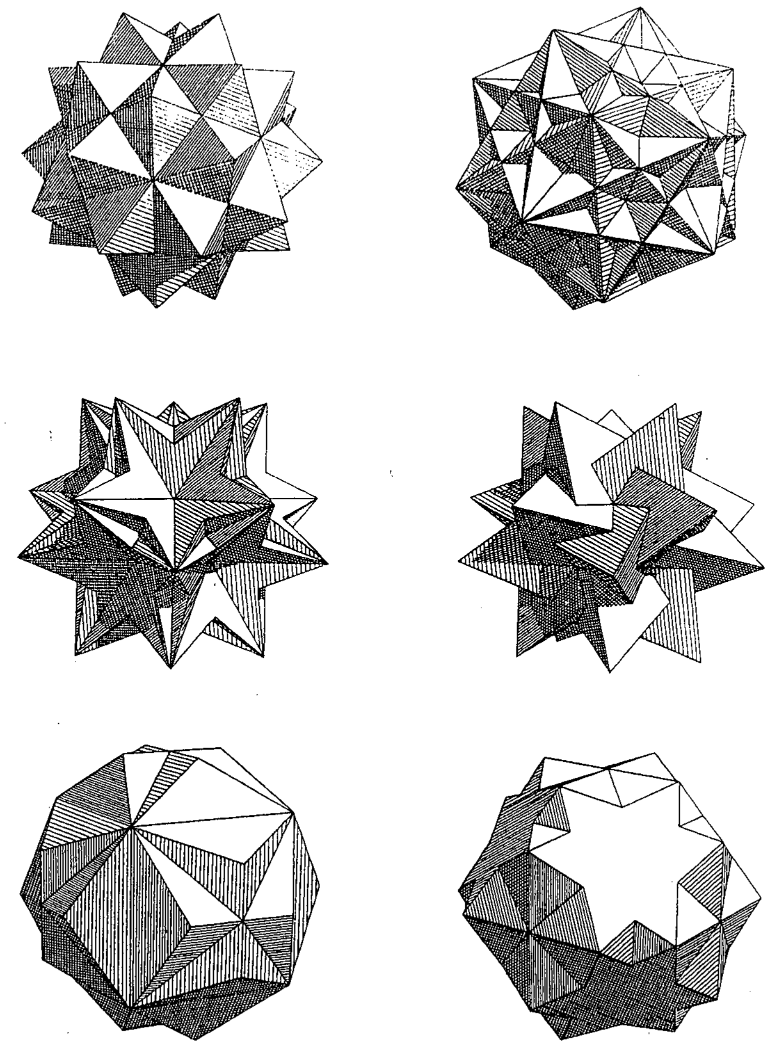


COMPOUND POLYHEDRA

a stretch of the imagination

The interrelationships on the previous page generate particularly beautiful compound polyhedra. Fix the position of an icosahedron, and octahedra can be placed around it in five different ways, giving the compound of five octahedra (*top left*). Similarly the cube within the dodecahedron, placed five different ways, generates the compound of five cubes (*top right*). The tetrahedron can be placed in the cube two different ways to give the compound of two tetrahedra shown on page 16. Replace each of the five cubes in the dodecahedron with two tetrahedra to give the compound of ten tetrahedra (*middle left*). Remove five of the tetrahedra from the compound of ten, to leave the compound of five tetrahedra (*middle right*). This occurs in two versions, right-handed, or *dextro,* and left-handed, or *laevo;* the two versions cannot be superimposed and are described as each other's *enantiomorphs.* Polyhedra or compounds with this property of "handedness" are referred to as *chiral.*

Returning to the cube and dodecahedron, and this time fixing the cube, there are two ways to place the dodecahedron around it. The result of both ways used simultaneously is the compound of two dodecahedra (*bottom left*). In the same way the octahedron and icosahedron pair gives the compound of two icosahedra (*bottom right*). Many other extraordinary compound polyhedra are possible; for example, Bakos's compound of four cubes is shown on the first page of this book.



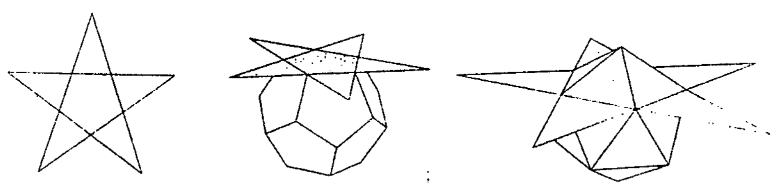
The Kepler Polyhedra

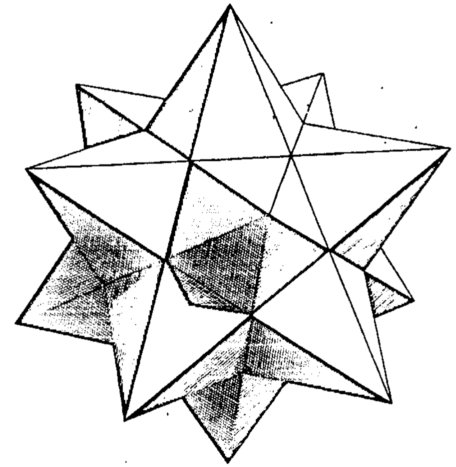
the stellated and great stellated dodecahedron

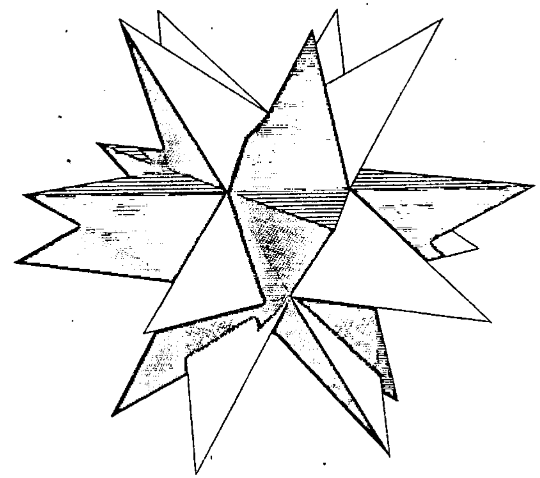
The sides of some polygons can be extended until they meet again; for example, the regular pentagon extends to form a five pointed star, or pentagram (*below*). This process is known as *stellation.* Johannes Kepler (1571–1630) proposed its application to polyhedra, observing the two possibilities of stellation by extending edges, and stellation by extending face planes. Applying the first of these (*below*) to the dodecahedron and icosahedron he discovered the two polyhedra illustrated opposite and named them the larger and smaller icosahedral hedgehogs.

Their modern names, the stellated dodecahedron (*opposite, top*) and the great stellated dodecahedron (*opposite, bottom*), reveal that these polyhedra are also two of the face stellations of the dodecahedron. Each is made of twelve pentagram faces, one with five, the other with three to every vertex. They have icosahedral symmetry.

Although its five sides intersect each other, the pentagram has equal edges and equal angles at its vertices and so can be considered a nonconvex regular polygon. Likewise, these polyhedra can be regarded as nonconvex regular polyhedra.





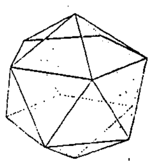
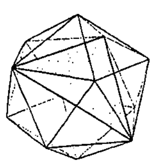
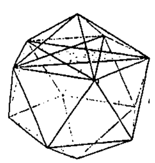
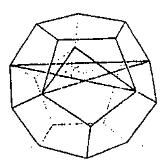


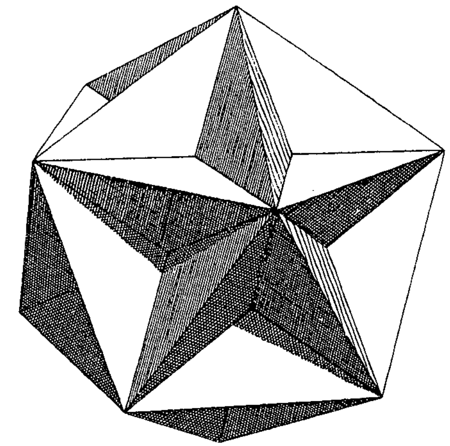
THE POINSOT POLYHEDRA

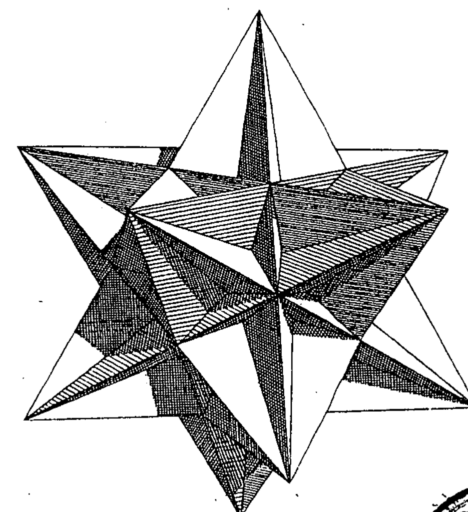
the great dodecahedron and the great icosahedron

Louis Poinsot (1777–1859) investigated polyhedra independently of Kepler. He rediscovered Kepler's two icosahedral hedgehogs and also discovered the two polyhedra shown here: the great dodecahedron (*opposite, top*) and the great icosahedron (*opposite, bottom*). Both of these polyhedra have five faces to a vertex, intersecting each other to give pentagram *vertex figures*. The great dodecahedron has twelve pentagonal faces and is the third stellation of the dodecahedron. The great icosahedron has twenty triangular faces and is one of an incredible fifty-nine possible stellations of the icosahedron, which also include the compounds of five octahedra and of five and ten tetrahedra.

A nonconvex regular polyhedron must have vertices arranged like one of the Platonic solids. Joining a polyhedron's vertices to form new types of polygon within it is known as *faceting*. The possibilities of faceting the Platonic solids produce the compounds of two and ten tetrahedra, the compound of five cubes, the two Poinsot polyhedra (*below left*) and the two Kepler star polyhedra (*below right*). The four Kepler-Poinsot polyhedra are therefore the only nonconvex regular polyhedra.





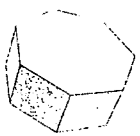
The Archimedean Solids

thirteen semiregular polyhedra

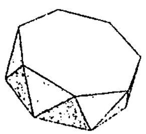
The thirteen Archimedean solids (*opposite*) are the subject of much of the rest of this book. Also known as the *semiregular polyhedra*, they have regular faces of more than one type, and identical vertices. They all fit perfectly within a sphere, with tetrahedral, octahedral, or icosahedral symmetry. Although their earliest attribution is to Archimedes (c. 287 B.C.–212 B.C.), Kepler seems to have been the first person since antiquity to describe the whole set of thirteen in his *Harmonices Mundi*. He further noted the two infinite sets of regular prisms and antiprisms (*below*), which also have identical vertices and regular faces.

Turn one octagonal cap of the rhombicuboctahedron by an eighth of a turn to obtain the pseudorhombicuboctahedron (*below*). Its vertices, while surrounded by the same regular polygons, are of *two* types relative to the polyhedron as a whole.

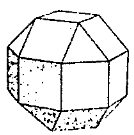
There are fifty-three semiregular nonconvex polyhedra, one example being the dodecadodecahedron (*below*). Together with the Platonic and Archimedean solids, and the Kepler-Poinsot polyhedra, they form the set of seventy-five uniform polyhedra.



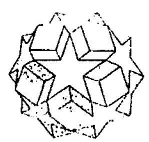
heptagonal prism



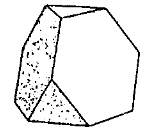
heptagonal antiprism



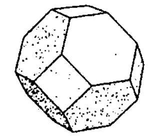
pseudo rhombicuboctahedron



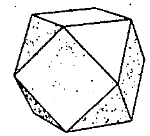
dodecadodecahedron



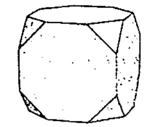
truncated tetrahedron



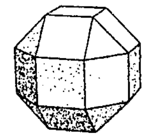
truncated octahedron



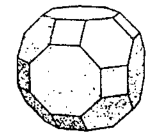
cuboctahedron



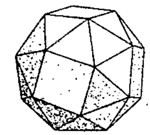
truncated cube



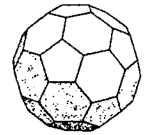
rhombicuboctahedron



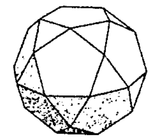
great rhombicuboctahedron



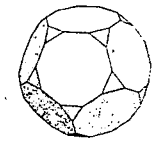
snub cube



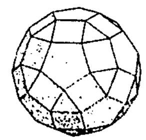
truncated icosahedron



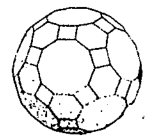
icosidodecahedron



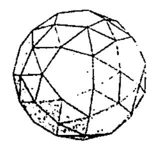
truncated dodecahedron



rhombicosidodecahedron



great rhombicosidodecahedron



snub dodecahedron

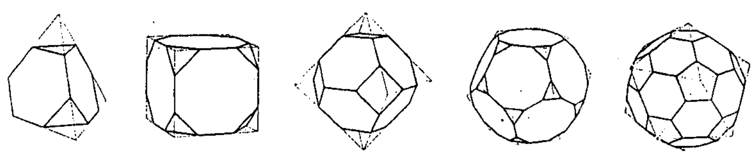
FIVE TRUNCATIONS

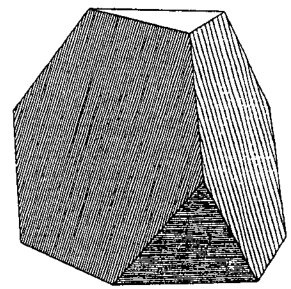
off with their corners!

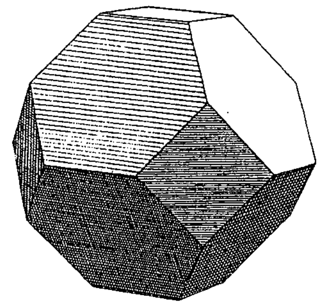
Truncate the Platonic solids to produce the five equal-edged Archimedean polyhedra shown here. These truncated solids are the perfect demonstration of the Platonic solids' vertex figures: triangular for the tetrahedron, cube, and dodecahedron; square for the octahedron; and pentagonal for the icosahedron. Each Archimedean solid has one circumsphere and one midsphere. They have an insphere for each type of face, the larger faces having the smaller inspheres touching their centers. Each truncated solid therefore defines four concentric spheres.

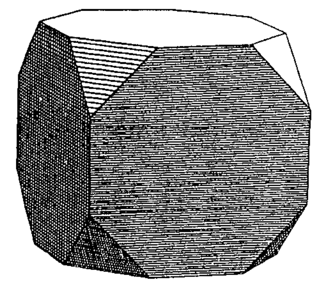
The five truncated solids can each sit neatly inside both their original Platonic solid and that solid's dual. For example, the truncated cube can rest its octagonal faces within a cube or its triangular faces within an octahedron.

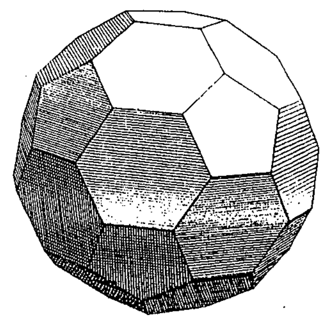
The truncated octahedron is the only Archimedean solid that can fill space with identical copies of itself, leaving no gaps. It also conceals a less obvious secret. Joining the ends of one of its edges to its center produces a central angle that is the same as the acute angle in the famous Pythagorean 3 : 4 : 5 triangle, beloved by ancient Egyptian masons for defining a right angle.

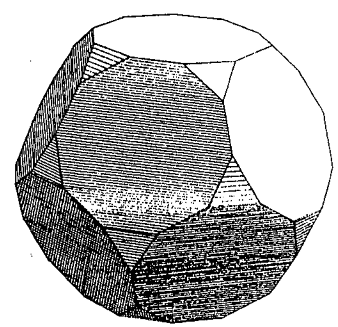












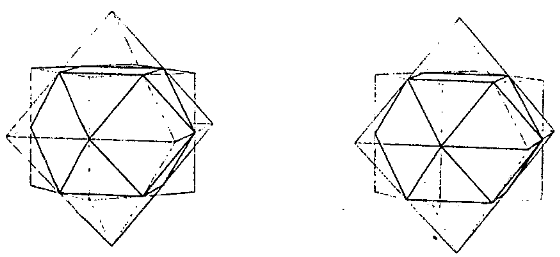
The Cuboctahedron

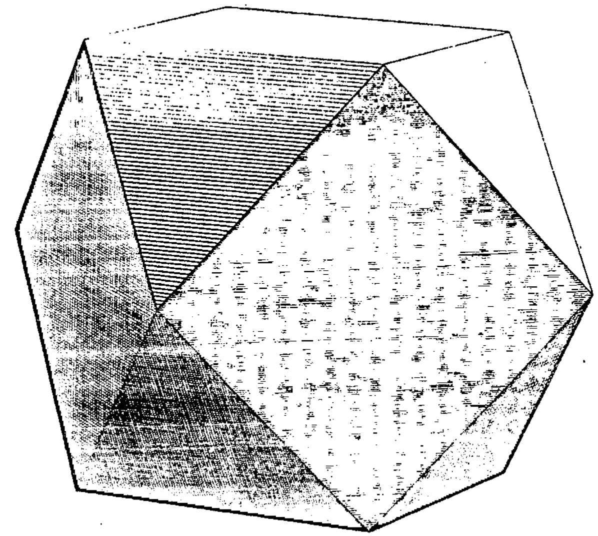
14 faces, 24 edges, 12 vertices

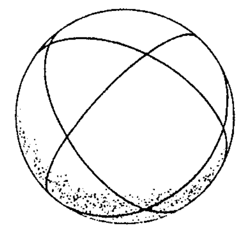
The cuboctahedron combines the six square faces of the cube with the eight triangular faces of the octahedron. It has octahedral symmetry. Joining the edge midpoints of either the cube or the octahedron traces out a cuboctahedron (*shown below as a stereogram pair*). According to Heron of Alexandria (10–75), Archimedes ascribed the cuboctahedron to Plato.

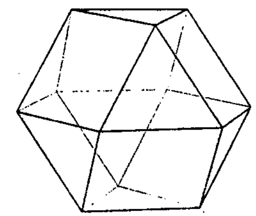
Quasiregular polyhedra such as the cuboctahedron are made of two types of regular polygon, each type being surrounded by polygons of the other type. The identical edges, in addition to defining the faces themselves, also define equatorial polygons. For example, the cuboctahedron's edges define four regular hexagons. The radial projections of quasiregular polyhedra consist entirely of complete great circles (*opposite, bottom left*).

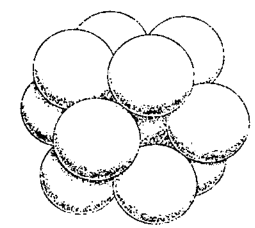
The maximum number of identical spheres that can fit around a central sphere of equal size is twelve. Arranged symmetrically so that their centers define the vertices of a cuboctahedron, they each touch four neighbors (*opposite, bottom right*).









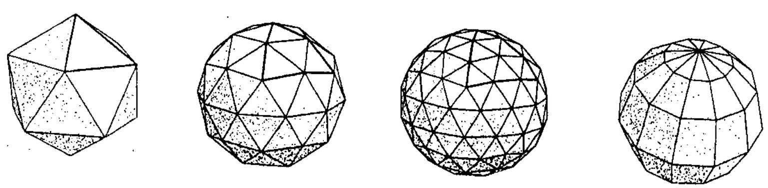


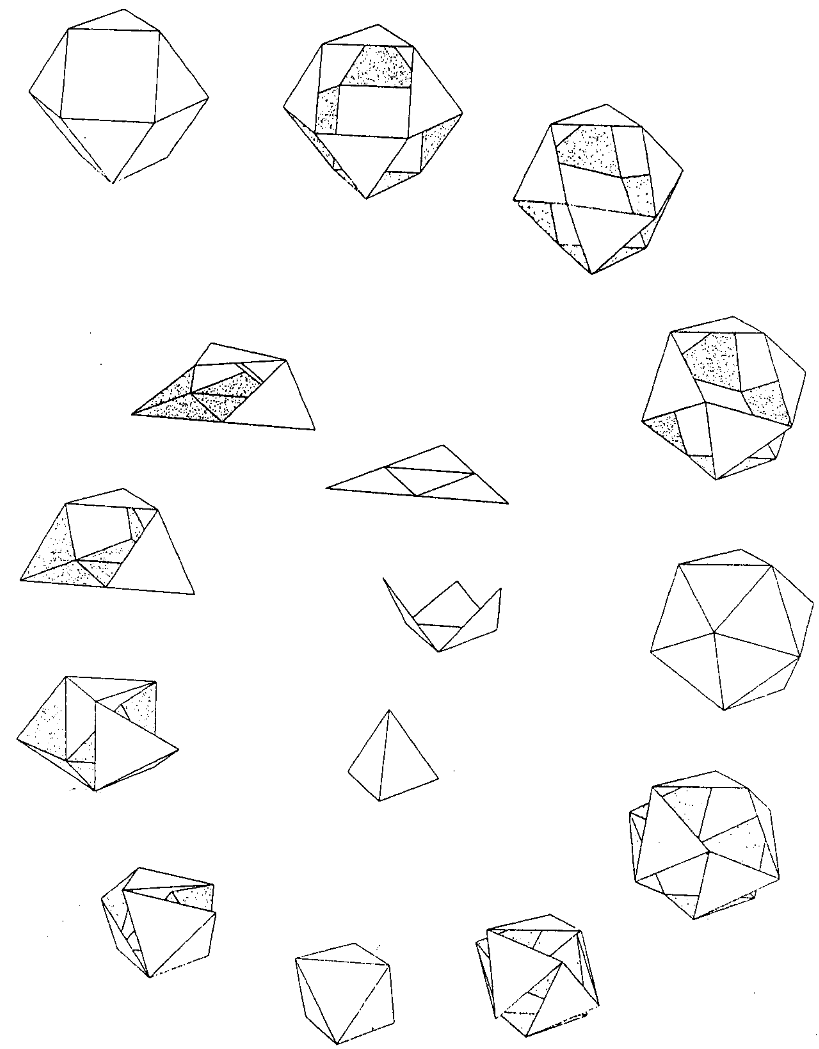
A Cunning Twist

and a structural wonder

Picture a cuboctahedron made of rigid struts joined at flexible vertices. This structure was named "the jitterbug" by R. Buckminster Fuller (1895–1983), and is shown opposite with the rigid triangular faces filled in for clarity. The jitterbug can be slowly collapsed in on itself in two ways so that the square "holes" become distorted. When the distance between the closing corners equals the edge length of the triangles, an icosahedron is defined. Continue collapsing the structure and it becomes an octahedron. If the top triangle is then given a twist the structure flattens to form four triangles that close up to give the tetrahedron.

Geodesic domes are another of Buckminster Fuller's structural discoveries. These are parts of geodesic spheres, which are formed by subdividing the faces of a triangular polyhedron, usually the icosahedron, into smaller triangles, and then projecting the new vertices outward to the same distance from the center as the original ones (*below*). A distant relative of the geodesic sphere is the popular Renaissance polyhedron of seventy-two sides known as Campanus's sphere (*below right*).





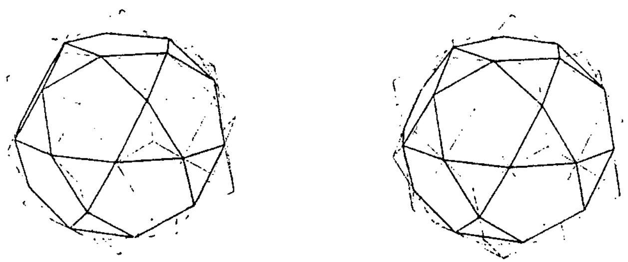
The Icosidodecahedron

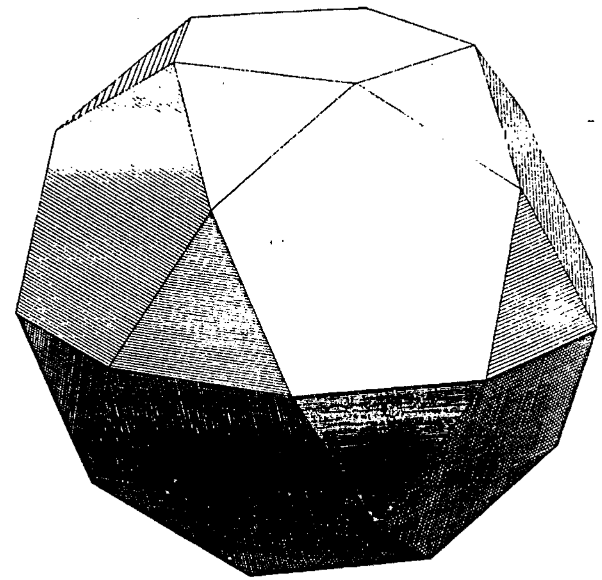
32 faces, 60 edges, 30 vertices

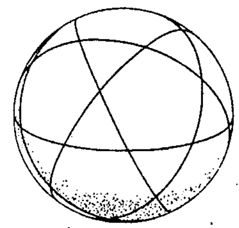
The icosidodecahedron combines the twelve pentagonal faces of the dodecahedron with the twenty triangular faces of the icosahedron. Joining the edge midpoints of either the dodecahedron or the icosahedron traces out the quasiregular icosidodecahedron (*both shown below as a stereogram pair*). Its edges form six equatorial decagons, giving a radial projection of six great circles (*opposite, bottom left*).

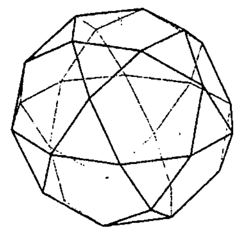
The earliest known depiction of the icosidodecahedron is by Leonardo Da Vinci (1452–1519) and appears in Fra Luca Pacioli's (1445–1517) *De Divina Proportione*. Appropriately this work's main theme is the golden ratio, which is perfectly embodied by the ratio of the icosidodecahedron's edge to its circumradius.

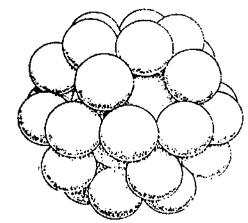
Defining the icosidodecahedron with thirty equal spheres leaves space for a large central sphere that is $\sqrt{5}$ (*see page 55*) times as large as the others (*opposite, bottom right*).











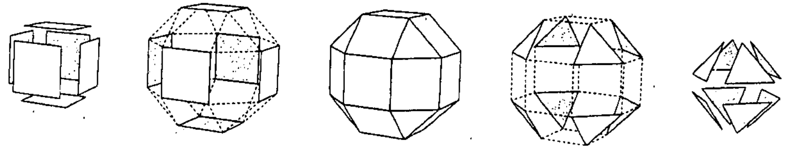
Four Explosions

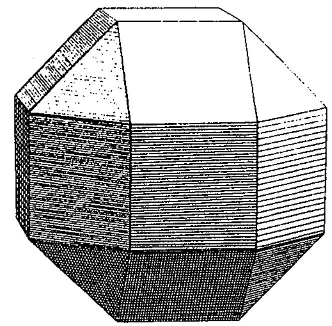
expanding from the center

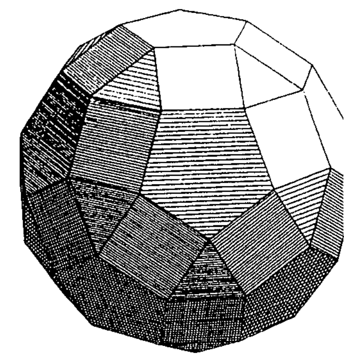
Exploding the faces of the cube or the octahedron outward until they are separated by an edge length (*below*) defines the rhombicuboctahedron (*opposite, top left*). The same process applied to the dodecahedron or icosahedron gives the rhombicosidodecahedron (*opposite, top right*). The octagonal faces of the truncated cube, or the hexagonal faces of the truncated octahedron, explode to give the great rhombicuboctahedron (*opposite, bottom left*). The decagonal faces of the truncated dodecahedron, or the hexagonal faces of the truncated icosahedron, explode to give the great rhombicosidodecahedron (*opposite, bottom right*).

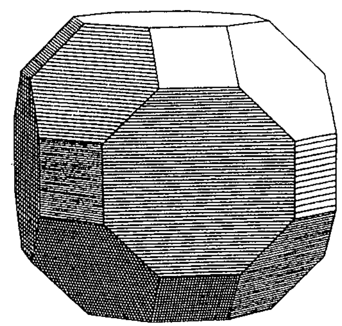
Kepler called the great rhombicuboctahedron a truncated cuboctahedron, and the great rhombicosidodecahedron a truncated icosidodecahedron. Truncating these polyhedra, however, does not leave square faces, but $\sqrt{2}$ and ϕ rectangles.

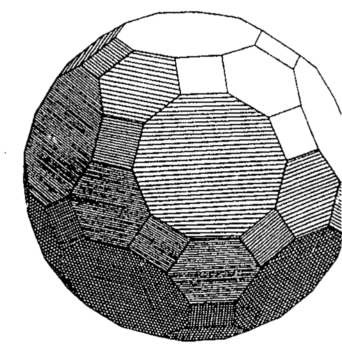
These four polyhedra have face planes in common with either the cube, octahedron, and rhombic dodecahedron (*see page 47*), or the icosahedron, dodecahedron, and rhombic triacontahedron (*see page 47*), hence the prefix "rhombi-" in their names.











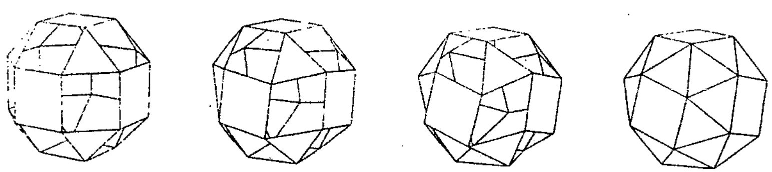
TURNING

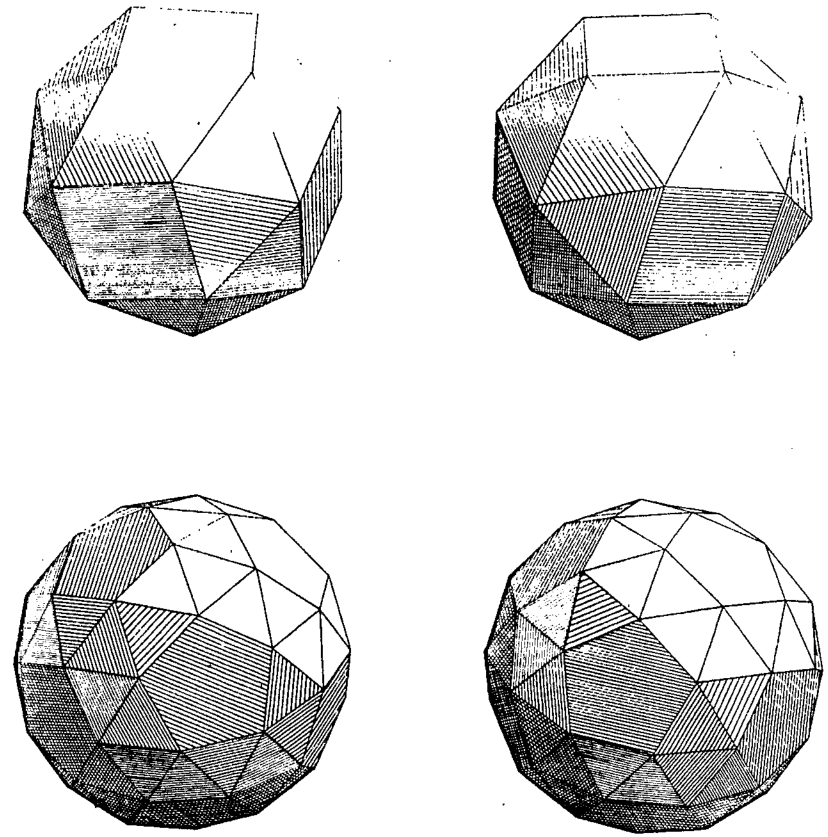
the snub cube and the snub dodecahedron

The name "snub cube" is a loose translation of Kepler's name *cubus simus*, literally "the squashed cube." Both the snub cube and the snub dodecahedron are chiral, occurring in dextro and laevo versions. Both versions are illustrated opposite with the dextro versions on the right. The snub cube has octahedral symmetry, and the snub dodecahedron has icosahedral symmetry. Neither has any mirror planes. Of the Platonic and Archimedean solids the snub dodecahedron is closest to the sphere.

The rhombicuboctahedron (*see page 43*) can be used to make a structure similar to the jitterbug (*see page 39*). Applying a twist to this new structure produces the snub cube (*below*). Twist one way to make the dextro version and the other to make the laevo. The corresponding relationship exists between the rhombicosidodecahedron and the snub dodecahedron.

The five Platonic solids have been truncated, combined, exploded, and twisted into the thirteen Archimedean solids. Three-dimensional space is revealing its order, complexity, and subtlety. What other wonders await?



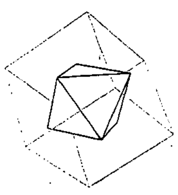


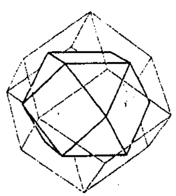
The Archimedean Duals

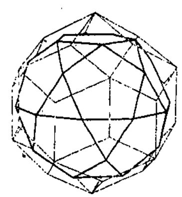
everything has its opposite

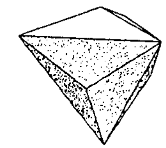
The duals of the Archimedean solids were first described as a group by Eugène Catalan (1814–1894) and are positioned opposite to correspond with their partners on page 33. To create the dual of an Archimedean solid, extend perpendicular lines from its edge midpoints, tangential to the solid's midsphere. These lines are the dual's edges, the points where they first intersect each other are its vertices. Archimedean solids have one type of vertex and different types of faces, their duals therefore have one type of face but different types of vertices.

The two quasiregular Archimedean solids, the cuboctahedron and the icosidodecahedron, both have rhombic duals that were discovered by Kepler. The Platonic dual pair compounds (*pages 16, 36, and 40*) define the face diagonals of these rhombic polyhedra, which are in the ratios √2 for the rhombic dodecahedron and ϕ for the rhombic triacontahedron. Kepler noticed that bees terminate their hexagonal honeycomb cells with three such √2 rhombs. He also described the three dual pairs involving quasiregular solids (*below*), where the cube is seen as a rhombic solid, and the octahedron as a quasiregular solid.

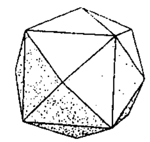




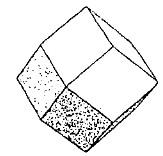




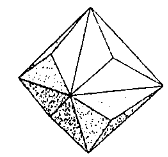
triakistetrahedron



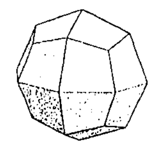
tetrakishexahedron



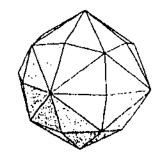
rhombic dodecahedron



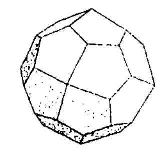
triakisoctahedron



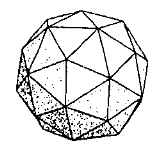
trapezoidal icositetrahedron



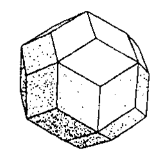
disdyakisdodecahedron



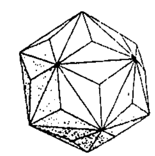
pentagonal icositetrahedron



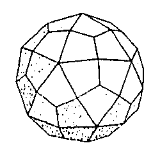
pentakisdodecahedron



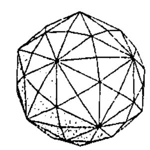
rhombic triacontahedron



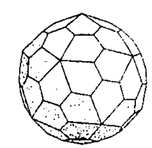
triakisicosahedron



trapezoidal hexecontahedron



disdyakistriacontahedron



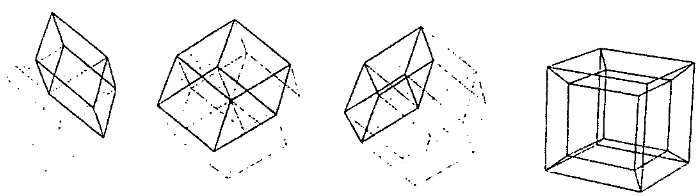
pentagonal hexecontahedron

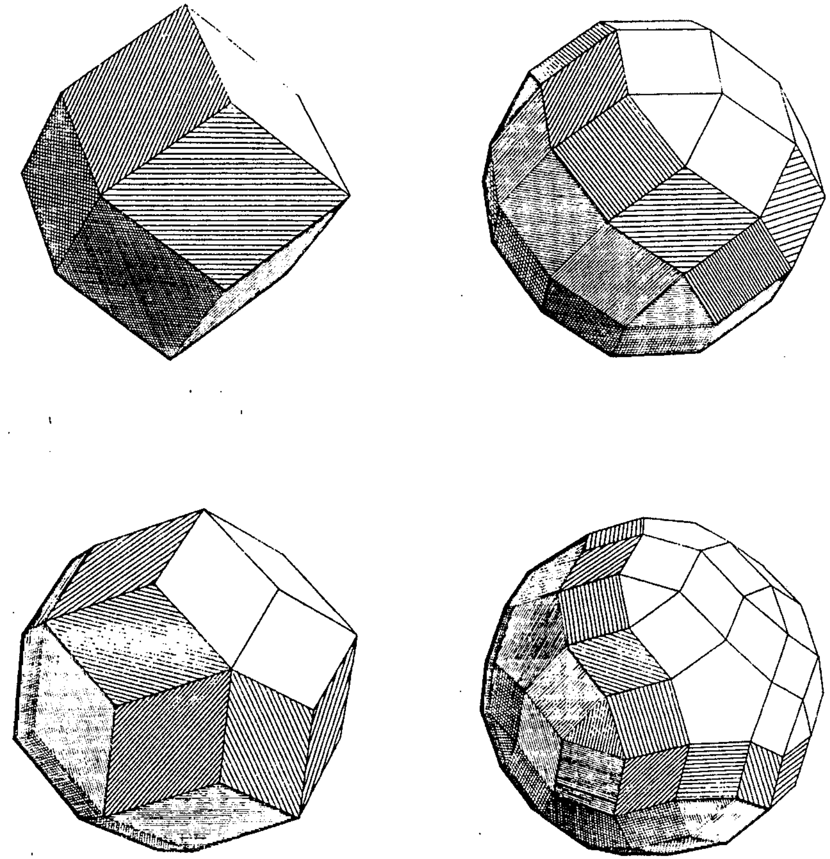
More Explosions

and unseen dimensions

Exploding the rhombic dodecahedron, or its dual the cuboctahedron, results in an equal edged convex polyhedron of fifty faces (*opposite, top right*). The exploded rhombic triacontahedron, or exploded icosidodecahedron, has one hundred and twenty-two faces (*opposite, bottom right*).

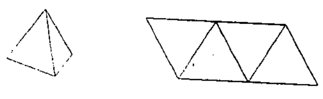
Ludwig Schläfi (1814–1895) proved that there are six regular four-dimensional *polytopes* (generalizations of polyhedra): the 5-cell made of tetrahedra; the 8-cell, or *tesseract,* made of cubes; the 16-cell made of tetrahedra; the 24-cell made of octahedra; the 120-cell made of dodecahedra; and the 600-cell made of tetrahedra. The rhombic dodecahedron is a three-dimensional shadow of the four-dimensional tesseract analogous to the hexagon as a two-dimensional shadow of the cube. In a cube two squares meet at every edge. In a tesseract three squares meet at every edge. Squares through the same edge define three cubes (*shaded below with an alternative tesseract projection*). Schläfi also proved that in five or more dimensions the only regular polytopes are the *simplex*, or generalized tetrahedron, the *hypercube*, or generalized cube, and the *orthoplex*, or generalized octahedron.



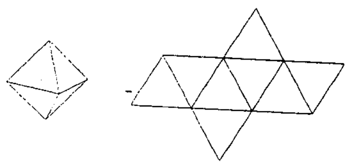


Flat-Packed Polyhedra

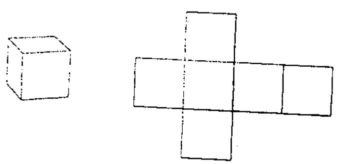
If a polyhedron is "undone" along some of its edges and folded flat, the result is known as its *net*. The earliest known examples of polyhedra presented this way are found in Albrecht Dürer's *Painter's Manual*, from 1525. The nets below are scaled such that if refolded the resulting polyhedra would all have equal circumspheres.



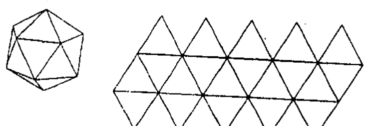
tetrahedron



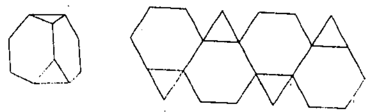
octahedron



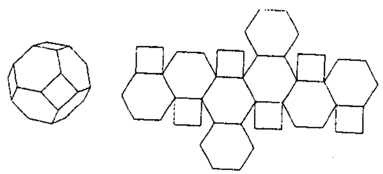
cube



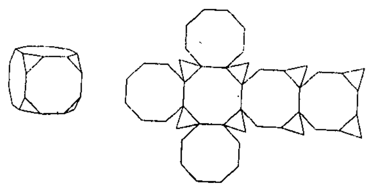
icosahedron



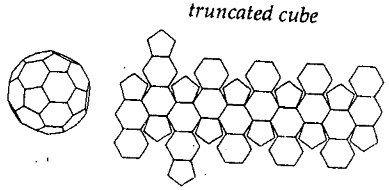
truncated tetrahedron



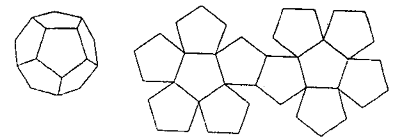
truncated octahedron



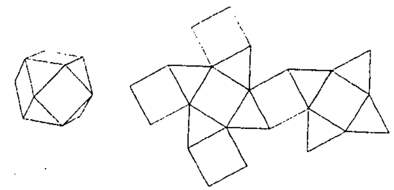
truncated cube



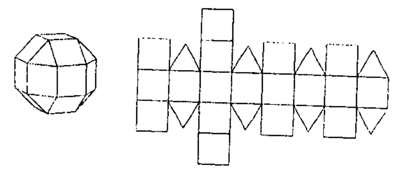
truncated icosahedron



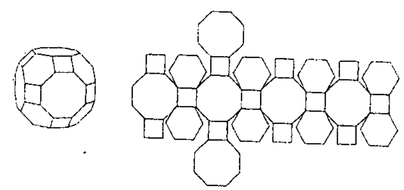
dodecahedron



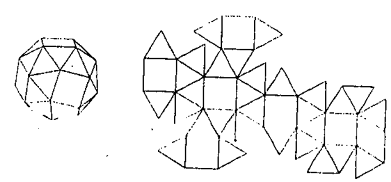
cuboctahedron



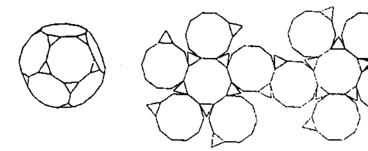
rhombicuboctahedron



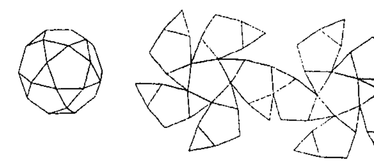
great rhombicuboctahedron



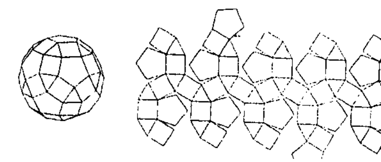
snub cube



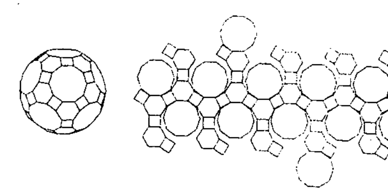
truncated dodecahedro



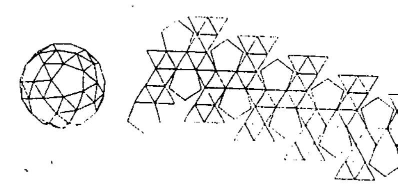
icosidodecahedron



rhombicosidodecahedron



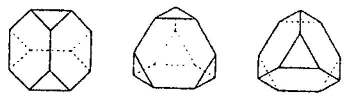
great rhombicosidodecahedr



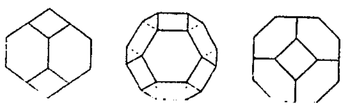
snub dodecahedron

ARCHIMEDEAN SYMMETRIES

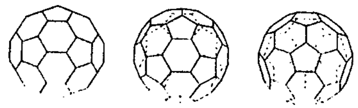
The diagrams below show the rotation symmetries of the Archimedean solids and the two rhombic Archimedean duals.



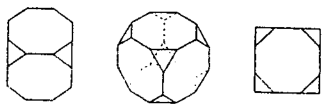
truncated tetrahedron



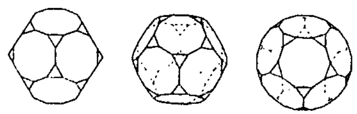
truncated octahedron



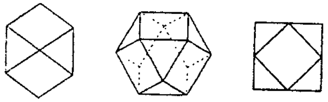
truncated icosahedron



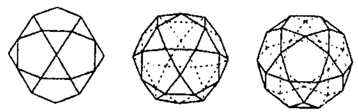
truncated cube



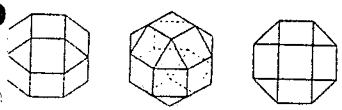
truncated dodecahedron



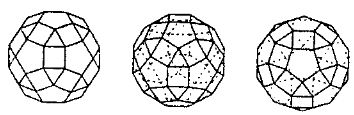
cuboctahedron



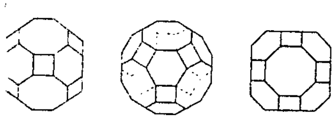
icosidodecahedron



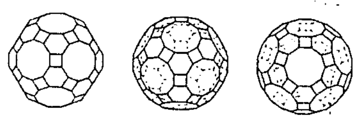
rhombicuboctahedron



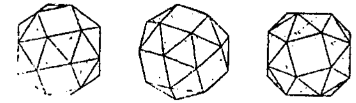
rhombicosidodecahedron



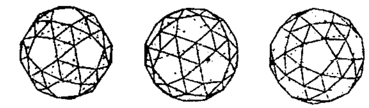
great rhombicuboctahedron



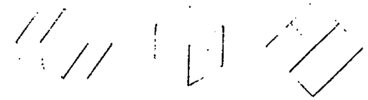
great rhombicosidodecahedron



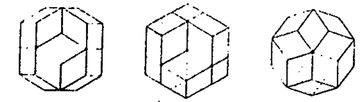
snub cube



snub dodecahedron



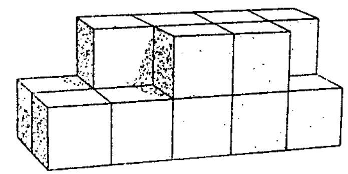
rhombic dodecahedron



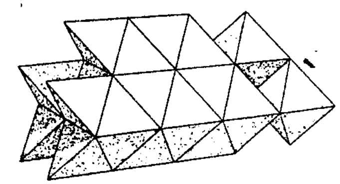
rhombic triacontahedron

THREE-DIMENSIONAL TESSELLATIONS

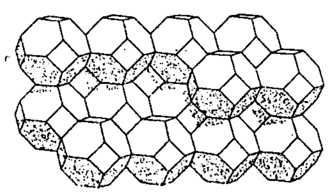
Of the Platonic solids only the cube can fill space with copies of itself and leave no gaps. The only other purely "Platonic" space filling combines tetrahedra and octahedra. One Archimedean solid, the truncated octahedron, and one Archimedean dual, the rhombic dodecahedron, are also space-filling polyhedra.



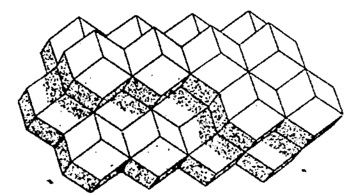
cubes



tetrahedra & octahedra

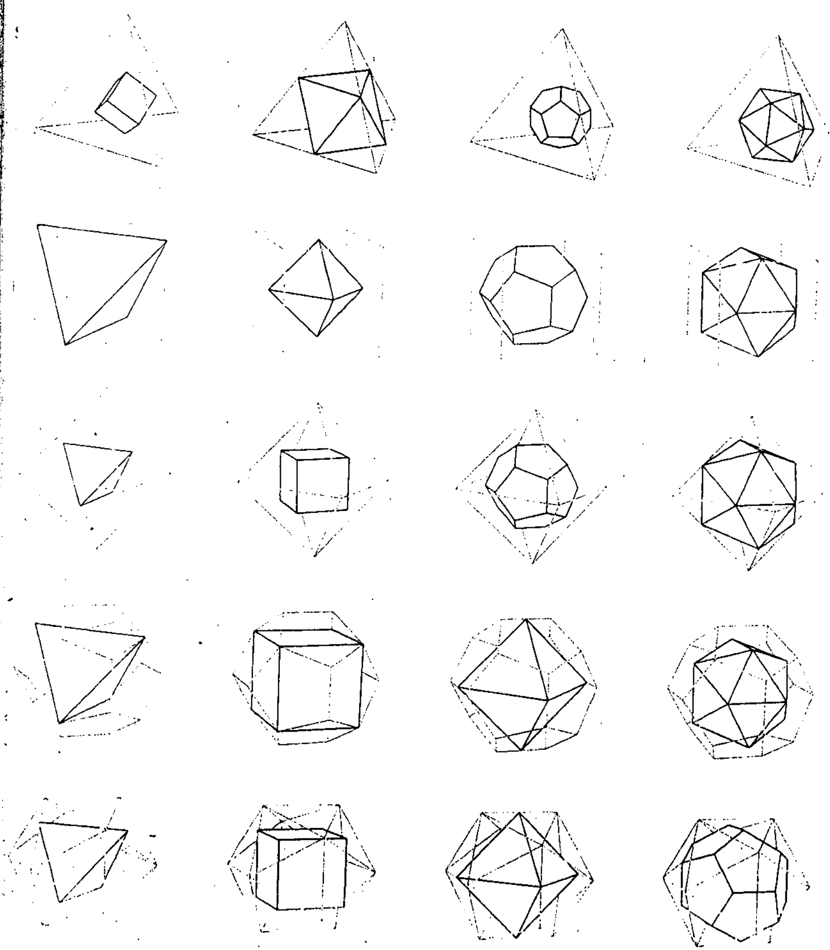


truncated octahedra



rhombic dodecahedra

Each Embracing Every Other



Expansions and Formulas

A recurring theme in the metric properties of the Platonic solids is the occurrence of th irrational numbers phi (ϕ), and the square roots $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$. They are surprisingl elegant when expressed as (infinitely) continued fractions:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} \qquad \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} \qquad \sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}} \qquad \sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}}$$

Their decimal expansions to twelve places, together with that of π are

$$\phi = 1.618033988750 \qquad \sqrt{2} = 1.414213562373 \qquad \sqrt{3} = 1.732050807569$$
$$\sqrt{5} = 2.236067977500 \qquad \pi = 3.141592653590$$

The table below gives volumes and surface areas for a sphere radius r, and Platonic solids, edge length s. Also included are the proportional pathways joining each vertex to every other in the Platonic solids.

	Volume	*Surface Area*	*Number of Pathways, Length*
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$	n/a
Tetrahedron	$\frac{\sqrt{2}}{12} s^3$	$\sqrt{3} s^2$	6 edges, s
Octahedron	$\frac{\sqrt{2}}{3} s^3$	$2\sqrt{3} s^2$	12 edges, s 3 axial diagonals, $\sqrt{2} s$
Cube	s^3	$6 s^2$	12 edges, s 12 face diagonals (inscribed tetrahedra), $\sqrt{2} s$ 4 axial diagonals, $\sqrt{3} s$
Icosahedron	$\frac{5}{6}\phi^2 s^3$	$5\sqrt{3} s^2$	30 edges, s 30 face diagonals, ϕs 6 axial diagonals, $\sqrt{(\phi^2+1)} s$
Dodecahedron	$\frac{\sqrt{5}}{2}\phi^4 s^3$	$3\sqrt{(25+10\sqrt{5})} s^2$	30 edges, s 60 face diagonals (inscribed cubes), ϕs 60 interior diagonals (inscr. tetrahedra), $\sqrt{2}\phi s$ 30 interior diagonals, $\phi^2 s$ 10 axial diagonals, $\sqrt{3}\phi s$

DATA TABLE

	Symmetry*	Vertices	Edges	Faces (total)	Faces (types)	Inradius*** / Circumradius	Midradius*** / Circumradius	Edge Length*** / Circumradius	Dihedral Angles****	Central Angle*****
Tetrahedron	Tetr.	4	6	4	4 triangles	0.3333333333	0.5773502692	1.6329931619	70°31'44"	109°28'16"
Cube	Oct.	8	12	6	6 squares	0.5773502692	0.8164965809	1.1547005384	90°00'00"	70°31'44"
Octahedron	Oct.	6	12	8	8 triangles	0.5773502692	0.7071067812	1.4142135624	109°28'16"	90°00'00"
Dodecahedron	Icos.	20	30	12	12 pentagons	0.7946544723	0.9341723590	0.7136441795	116°33'54"	41°48'37"
Icosahedron	Icos.	12	30	20	20 triangles	0.7946544723	0.8506508084	1.0514622242	138°11'23"	63°26'06"
Stellated Dodecahedron	Icos.	12	30	12	12 pentagrams	0.4472135955	0.5257311121	1.7013016167	116°33'54"	116°33'54"
Great Dodecahedron	Icos.	12	30	12	12 pentagons	0.4472135955	0.8506508084	1.0514622242	63°26'06"	63°26'06"
Great Stellated Dodecahedron	Icos.	20	30	12	12 pentagrams	0.1875924741	0.3568220898	1.8683447179	63°26'06"	138°11'23"
Great Icosahedron	Icos.	12	30	20	20 triangles	0.1875924741	0.5257311121	1.7013016167	41°48'37"	116°33'54"
Cuboctahedron	Oct.	12	24	14	8 triangles 6 squares	0.8164965809 0.7071067812	0.8660254038	1.0000000000	125°15'52"	60°00'00"
Icosidodecahedron	Icos.	30	60	32	20 triangles 12 pentagons	0.9341723590 0.8506508084	0.9510565163	0.6180339887	142°37'21"	36°00'00"
Truncated Tetrahedron	Tetr.	12	18	8	4 triangles 4 hexagons	0.8703882798 0.5222329679	0.9045340337	0.8528028654	70°31'44" 109°28'16"	50°28'44"
Truncated Cube	Oct.	24	36	14	8 triangles 6 octagons	0.9458621650 0.6785983445	0.9596829823	0.5621692754	90°00'00" 125°15'52"	32°39'00"
Truncated Octahedron	Oct.	24	36	14	6 squares 8 hexagons	0.8944271910 0.7745966692	0.9486832981	0.6324555320	109°28'16" 125°15'52"	36°52'12"
Truncated Dodecahedron	Icos.	60	90	32	20 triangles 12 decagons	0.9809163757 0.8385051474	0.9857219193	0.3367628118	116°33'54" 142°37'21"	19°23'15"
Truncated Icosahedron	Icos.	60	90	32	12 pentagons 20 hexagons	0.9392336205 0.9149583817	0.9794320855	0.4035482123	138°11'23" 142°37'21"	23°16'53"
Rhombicuboctahedron	Oct.	24	48	26	8 triangles 18 squares	0.9108680249 0.8628562095	0.9339488311	0.7148134887	135°00'00" 144°44'08"	41°52'55"
Great Rhombicuboctahedron	Oct.	48	72	26	12 squares 8 hexagons 6 octagons	0.9523198087 0.9021230715 0.8259425910	0.9764509762	0.4314788105	125°15'52" 135°00'00" 144°44'08"	24°55'04"
Rhombicosidodecahedron	Icos.	60	120	62	20 triangles 30 squares 12 pentagons	0.9659953695 0.9485360199 0.9245941063	0.9746077624	0.4478379596	148°16'57" 153°56'33" 159°05'41"	25°52'43"
Great Rhombicosidodecahedron	Icos.	120	180	62	30 squares 20 hexagons 12 decagons	0.9825566436 0.964797963 0.9049441875	0.9913166895	0.2629921751	142°37'21" 148°16'57" 159°05'41"	15°06'44"
Snub Cube	Oct.-**	24	60	38	32 triangles 6 squares	0.9029870683 0.8503402074	0.9281913780	0.7442063312	142°59'00" 153°14'05"	43°41'27"
Snub Dodecahedron	Icos.-**	60	150	92	80 triangles 12 pentagons	0.9634723304 0.9188614921	0.9727328506	0.4638568806	152°55'48" 164°10'31"	26°49'17"

* *Symmetries: Tetrahedral: 4 x 3-fold axes, 3 x 2-fold, 6 mirror planes. Octahedral: 3 x 4-fold axes, 4 x 3-fold, 6 x 2-fold, 9 mirror planes. Icosahedral: 6 x 5-fold axes, 10 x 3-fold, 15 x 2-fold, 15 mirror planes.*

** *The snub solids have no mirror planes.*

*** *From the polyhedron's center the inradius is measured to the various face-centers, the midradius to the edge midpoints, and the circumradius to vertices.*

**** *In Archimedean solids the larger dihedral angles are found between smaller pairs of faces.*

***** *The central angle is the angle formed at the center of a polyhedron by joining the ends of an edge to that center.*

FURTHER READING

If you have enjoyed this Wooden Book, others in the series that may be of interest include *Sacred Geometry* by Miranda Lundy and *Useful Mathematical & Physical Formulæ* by Matthew Watkins.

For those looking for more things polyhedral, Keith Critchlow's *Order In Space* (Thames & Hudson) and Peter R. Cromwell's *Polyhedra* (Cambridge) are both highly recommended. H. S. M. Coxeter's *Regular Polytopes* (Dover) is the classic twentieth-century mathematical text on the subject, and Norman Johnson's forthcoming *Uniform Polytopes* (Cambridge) promises to become an indispensable addition to the literature. Those with access to a manuscript library are well advised to seek out Wenzel Jamnitzer's *Perspectiva Corporum Regularium* (1568).

For those wishing to make models, Magnus J. Wenninger's *Polyhedron Models* (Cambridge), *Dual Models* (Cambridge), and *Spherical Models* (Dover) cover their respective areas very thoroughly. *Shapes, Space and Symmetry* by Alan Holden (Dover) is also good. A range of cut-out-and-make-polyhedra books

Gec

1,000

It can